Arrow's Consideration of Ties and Indifference

by

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Abstract

In "Social Choice and Individual Values" Kenneth Arrow (1951) asserts that, in his model of social and individual choice, ties are considered. However, as this paper makes clear only ties among alternatives and not ties among preference orderings are taken into account although Arrow claims to consider ties between binary orderings, at least, in Axiom 1 using R, the preference or indifference relation. Since he defines indifference as the logical ANDing of the elements of a tie, Arrow's treatment of ties is confined to the inclusion of indifference in a preference ordering. Moreover, Arrow clearly intends for the specification of individual and social preference orderings to be made in terms of P (the preference relation) and I (the indifference relation). Therefore, R should be defined in terms of them and not the other way around. An examination of Arrow's axioms, definitions and conditions reveals a number of inconsistencies and errors involving the use of R. There are at least three different interpretations of R used by Arrow at different times giving rise to at least two different models. These models are both examined and the axioms, definitions and conditions are clarified for each. If R (the preference or indifference operator) information is primary, then P (preference) information has been abstracted from and hence cannot be extracted from the data and the conditions must be stated without reference to P. If P and I (indifference) information is primary and R derivative, then the conditions as stated are not quite correct and must be modified. The implications of the inclusion of ties in both models are examined. If ties are included, Arrow's conditions must be rewritten in a more general manner. The inclusion of ties provides for the existence of the Social Welfare Function (SWF) and solutions are presented for both models for the case of three alternatives.

Introduction

In "Social Choice and Individual Values" Kenneth Arrow (1951) postulates 2 axioms and 5 conditions which a Social Welfare Function (SWF) should meet in order to be considered rational and ethical and then goes on to show that such a SWF doesn't exist. He assumes a population of n voter/consumers who specify their preferences among m alternatives using the R (preference or indifference) operator. This then constitutes Arrow's model. In order to disprove Arrow's result, it would be necessary to substantially retain his model (although it can be shown to be somewhat arbitrary, incomplete and incorrect) or else a result is proven that applies to some other model and the validity of Arrow's result remains. However, to the extent that Arrow's model is undefined, self-contradicting or loose, it is possible to add, correct and tighten in such a way as to remain within his model. Arrow seems to provide for the inclusion of ties in a general way in Axiom 1, but it is clear from the later context that ties are only included in his analysis by means of the indifference operator I which provides for ties between alternatives. Ties among orderings are not included.

The primary relation in Arrow's model, R, means "preferred or indifferent". The statement "x is preferred or indifferent to y" is symbolized by xRy. Each individual voter/consumer is numbered so that it is possible to speak of the i^{th} voter/consumer whose choice between x and y is symbolized as xR_iy. If there are m alternatives in a set T, each individual expresses his preference ordering as $R_i = x_1R_ix_2 \ldots x_{j-1}R_ix_j \ldots x_{m-1}R_ix_m$, $x_j \in T$. Similarly, society's choice is given without the subscript i

i.e. $R = x_1Rx_2 \ldots x_{j-1}Rx_j \ldots x_{m-1}Rx_m$. Each individual makes an ordering; the totality of all possible individual orderings represents the domain. Then the SWF transforms each domain point into a social ordering which constitutes an element of the range.

Another issue in Arrow's model is whether or not the individuals and society have exactly the same choices available to them. Is the domain composed of exactly the same orderings as is the range? Arrow indicates that the social and individual orderings do not consist of exactly the same set. For individuals, "the chooser considers in turn all possible pairs of alternatives, say x and y, and for each such pair he makes one and only one of three decisions: x is preferred to y, x is indifferent to y, or y is preferred to x. The decisions made for different pairs are assumed to be consistent with each other, so, for example, if x is preferred to y and y to z, then x is preferred to z; similarly, if x is indifferent to y and y to z, then x is indifferent to z." For society the ordering over all alternatives without breaking them down by pairs is specified by the SWF. According to Arrow's definition: "By a SWF will be meant a process or rule which, for each set of individual orderings $R_1, ..., R_n$ for alternative social states (one ordering for each individual), states a corresponding social ordering of alternative social states, R." The choice between x and y by society is not simply a function of the individual choices between x and y but of the totality of individual choices according to the definition. However, the social ordering can be broken down into binary form due to the fact that it is transitive. For example, aPbIcPd can be broken down to the following: aPb, aPc, aPd, bIc, bPd, cPd, but aPb is not necessarily a function just of all of the aR_ib's. Note that with respect to the decomposition of the m^{th} stage social ordering into binary components, these components are not necessarily the same as the 2^{nd} stage social orderings. For example, if $aP^4bI^4cP^4d$ is the 4^{th} stage social ordering, aP^4b is not necessarily the same as the second stage ordering which might be bP^2a where P^k represents the k^{th} stage preference operator. This is all in accordance with Arrow's definition.

If ties are taken into account, then an individual might choose $\{xP_iy, xP_iy\}$ to indicate that he is equally divided between the two orderings as opposed to being indifferent between them. Society might choose $\{xPy, xPy\}$ to indicate that society is equally divided between the two orderings as opposed to being indifferent between them which could be indicated xIy. In this way the indifference operator represents a *relation* between alternatives. A tie represents an *equal weighting* of orderings. In general,

$R^m_i = \{(x_1R^m_ix_2 \ldots x_{j-1}R^m_ix_j \ldots x_{m-1}R^m_ix_m)^1, (x_1R^m_ix_2 \ldots x_{j-1}R^m_ix_j \ldots x_{m-1}R^m_ix_m)^2, \ldots, (x_1R^m_ix_2 \ldots x_{j-1}R^m_ix_j \ldots x_{m-1}R^m_ix_m)^p\}$, $x_j \in T$ where there are p elements of the tie. Similarly,

$R^m = \{(x_1R^mx_2 \ldots x_{j-1}R^mx_j \ldots x_{m-1}R^mx_m)^1, (x_1R^mx_2 \ldots x_{j-1}R^mx_j \ldots x_{m-1}R^mx_m)^2, \ldots, (x_1R^mx_2 \ldots x_{j-1}R^mx_j \ldots x_{m-1}R^mx_m)^p\}$. For purposes of simplification, one might choose to eliminate the indifference operator and consider only the preference operator and ties or one might choose, in the interest of rationality, to restrict individual orderings from including ties. In general, one can consider both preference and indifference operators and ties for both individuals and society.

Axiom 1 and Axiom 2

Arrow chooses as his primary relationship R because it is "slightly more convenient." Accordingly, P and I are derivative relationships. He demands in Axiom 1 that any two alternatives be comparable:

Axiom 1: *For all x and y, either xRy or yRx.*

He states: "Note also that the word 'or' in the statement of Axiom 1 does not exclude the possibility of both xRy and yRx. That word merely asserts that at least one of the two events must occur; both may." One assumes that Axiom 1 applies both to individuals and society. Therefore, the i^{th} individual would specify one of the following: 1) xR_iy; 2) yR_ix; 3) {xR_iy, yR_ix} where {xR_iy, yR_ix} is notation that indicates that "both [events occur]." It is obvious that, when both events occur, there is a tie. Society would likewise specify 1) xRy; 2) yRx; 3) {xRy, yRx}

It is also obvious that to Arrow a tie and an indifference amount to the same thing. He goes on to say "The adjective 'weak' refers to the fact that the ordering does not exclude indifference [i.e.] Axioms I and II do not exclude the possibility that for some distinct x and y, both xRy and yRx. A strong ordering, on the other hand, is a ranking in which no ties are possible." This would only be true if ties between orderings are excluded. Consider the strong ordering P and a situation in which half the individuals specify xP_iy and half specify yP_ix. Might not society then conclude that there is a tie between the two orderings xPy and yPx? Since Arrow considers a tie, in fact, to be identical to an indifference, he doesn't consider this possibility. But a tie is more general than an indifference. A tie between two alternatives represents an indifference between the alternatives whereas a tie between the orderings of two alternatives represents an even division between those two orderings. To say that society is indifferent between x and y is different from saying that society is evenly divided between xPy and yPx.

Arrow defines an indifference as a tie:

Definition 2: *xIy means xRy and yRx.*

Now it is not perfectly clear that this definition is identical to the case in Axiom 1 in which "both [xRy and yRx] occur." I would argue that the "and" of Definition 2 is the logical "and" which I symbolize as AND whereas, when "both xRy and yRx occur", their occurrence *need not* be connected by the logical AND, and a tie exists between xRy and yRx. For an individual or society to specify {xRy, yRx} is *heuristically* different from specifying xIy. Heuristically, {xRy, yRx} means that an individual or society can't make up his/their mind between the two possibilities, or gives equal weighting to the two possibilities or each of the two possibilities is equally as valid as the other whereas xIy means that the individual or society has decided definitely that either alternative has the same value, he/they "couldn't care less" between the two alternatives. In the tie situation the preference information has not been deleted whereas in the indifference situation it has. The tie situation means that the individual or society gives equal weighting to the two possibilities whereas the indifference situation means that the individual or society can't distinguish between them. Logically, however, according to Arrow's definition, an indifference and a tie are the same.

Arrow's axiom of transitivity states the following:

Axiom 2: *For all x, y, and z, xRy and yRz imply xRz.*

What this axiom effectively does is to exclude certain cases from consideration both by individuals and society on the grounds that it is illogical for an individual or society to prefer or be indifferent between x to/and y, prefer or be indifferent between y to/and z, and prefer or be indifferent between z to/and x. For the sake of completeness, Arrow should have listed the transitivity rules when ties occur following Axiom 1. Those are the following:

For all x, y, and z, if {xRy, yRx} and yRz, then xRz.
For all x, y, and z, if xRy and {yRz, zRy}, then xRz.
For all x, y, and z, if {xRy, yRx} and {yRz, zRy}, then {xRz, zRx}.

Arrow clearly sets out the case for a tie between xRy and yRx in Axiom 1, but neglects to prescribe the appropriate transitivity rules in Axiom 2.

Various Interpretations of R, P and I

There are several ways to interpret the relationships between R and P and I. On the one hand R ordering information can be considered primary and P and I ordering information derived from it by means of a logical function. According to Arrow's Axioms and Definition of a SWF, it would seem that this is his approach. On the other hand P and I ordering information can be considered primary and R ordering information derived from them by means of a logical function. Thirdly, the operator R can be considered a shorthand stand-in for P or I [i.e.] it must be replaced by the known operator (P or I) wherever it occurs. This would be the same as the R *operator* being defined as the P operator or the I operator.

P and I Derivable from R

In Definition 1, Arrow defines the P relationship in terms of the primary relationship R as follows:

Definition 1: *xPy is defined to mean not yRx.*

I is defined as follows:

Definition 2: *xIy means xRy and yRx.*

Also Arrow states the following:

Lemma 1(e): *For all x and y, either xRy or yPx.*

However, lemma 1(e) and axiom 1, which is restated here, are in conflict:

Axiom 1: *For all x and y, either xRy or yRx.*

In fact lemma 1(e), axiom 1 and definition 1 cannot all be true. One cannot have lemma 1(e) and axiom 1 both true unless yRx = yPx. The problem has to do with definition 1. According to axiom 1, one of the following must be true: xRy, yRx, {xRy, yRx}. Therefore, if yRx is not true (NOT yRx is true), then either xRy or {xRy, yRx} must be true—not xPy as Arrow states in lemma 1(e). Definition 1 overrides the implications of axiom 1. However, perhaps definition 1 can be salvaged by writing it as follows:

Definition 1': *xPy is defined to mean xRy and not yRx.*

or, alternatively,

Definition 1'': *xPy is defined to mean xRy and not {xRy, yRx}.*

Arrow's definition of a SWF is as follows:

Definition 4: *By a SWF will be meant a process or rule which, for each set of individual orderings $R_1, ..., R_n$ for alternative social states (one ordering for each individual), states a corresponding social ordering of alternative social states, R.*

According to Arrow's definition 4, one would assume that each of the individuals state their preferences in the form R_i = xRyRz, for example, and the SWF states society's preference in the form R = xRyRz, for example. Therefore, P and I are not expressed but implied by and/or derived from R_i and R. This would be logically consistent if each individual is required to specify his orderings as follows:
{xR_iy, NOT yR_ix}, {xR_iz, NOT zR_ix}, {yR_iz, zR_iy} for xP_iyI_iz. But according to the manner of specification demanded by Definition 4, the information specified both by individuals and society would be incomplete and not sufficient for deriving P and I information.

It is clear that Arrow intends for individuals to express their preferences as $^{e.g.}$ $x_1R_ix_2R_i...x_jR_ix_{j+1}R_i...x_{m-1}R_ix_m$ where the set of m alternatives is {x_1, x_2, ... x_m} and R_i is the preference relationship for the i^{th} individual. Arrow clearly intends that P and I information be derived from the primary R information, but, as we have seen, P information is not derivable from a specification of just xRy and, as we shall see shortly, I information is only partially derivable. Heuristically, if an individual states xR_iy, then we know that i prefers or (is indifferent between) x to/and y. What we don't know is whether he actually prefers or is actually indifferent between x and y. Let's say the reality in i's mind is xP_iy. Then xR_iy is a true statement for him. But let's say the reality in i's mind is xI_iy. Then xR_iy is also a true statement; but yR_ix is an equally valid statement. If i can't make up his mind between xP_iy and yP_ix, then {xR_iy, yR_ix} would be a true statement. In other words, if he knows he's *not* indifferent, but equally divided between xP_iy and yP_ix, then he would choose the tie as his choice. This casts doubt on Arrow's definition of indifference as a tie. If i is truly indifferent, he has the option of expressing xR_iy or yR_ix. However, if we *require* i to express an indifference as {xR_iy, yR_ix} and a preference as {xR_iy, NOT yR_ix}, then there is no ambiguity as to what i actually means. Otherwise, i would have 3 ways to express indifference: xR_iy, yR_ix, and {xR_iy, yR_ix}. In this interpretation, Arrow's definition of indifference as xR_iy AND yR_ix, would tell us only *some* of the indifferences and not those in which an individual expressed an indifference as xR_iy or yR_ix. By the same token, if xP_iy, then i could *only* express xR_iy and not either yR_ix or $\{xR_iy, yR_ix\}$ in order to be logically unambiguous. Therefore, it can be concluded that some I information can be derived from the normal specification of an R ordering, but no P information, and this conclusion holds both heuristically and logically.

R Derivable form P and I

Despite the fact that Arrow specifies R first and derives P and I from R and also specifies his definition of a SWF in terms of R, his real primary values, as the subsequent development of his exposition shows, are P and I and his intention is that individual and social orderings are specified in terms of P and I, not R. The model under these assumptions is the following:

Axiom 1': For all x and y, one and only one of the following must be true: xPy, yPx or xIy.

Axiom 2':

1) For all x, y and z, if xPy and yPz, then xPz;
2) For all x, y and z, if xPy and yIz, then xPz;
3) For all x, y and z, if xIy and yPz, then xPz;
4) For all x, y and z, if xIy and yIz, then xIz.

Definition 1': xRy is defined to mean xPy EOR xIy.
where EOR is the exclusive OR.

Definition 4': *By a SWF will be meant a process or rule which, for each set of individual orderings $Q_1, ..., Q_n$ for alternative social states (one ordering for each individual), states a corresponding social ordering of alternative social states, Q, where $Q_i = x_1Q^1{}_ix_2Q^2{}_i \dots x_jQ^j{}_ix_{j+1}Q_j^{j+1}{}_i \dots x_{m-1}Q_i^{m-1}{}_ix_m$, and $Q_i^k = P_i$ or I_i, $Q = x_1Q^1x_2Q^2 \dots x_jQ^jx_{j+1}Q^{j+1} \dots x_{m-1}Q^{m-1}{}_ix_m$ and $Q^k = P$ or I.*

Now in this model ties can be included or excluded. It is clear that Arrow's intention is that they be excluded except for ties among alternatives indicated by the indifference operator. Since he doesn't take ties among orderings into account, his theory is essentially correct although an examination of his 5 conditions shows some minor errors if this model is used. Ties among orderings in this model would be of the form, for example, {xPyIz, yIzIx, zPyPx}.

R as Shorthand for P or I

There is no logical relationship among R, P and I, but R is simply replaced in all expressions by P or I similar to Definition 4'. In some of his development, namely, Definition 4, Arrow seems to have used this interpretation. Definition 4' with Q replaced by R would be the correct specification for this interpretation of the relationship among R, P and I if there is no logical relationship among R, P and I. The *operator* R is equal to the *operator* P or the *operator* I as opposed to the *relation* xRy being equal to the *relation* xPy EOR the *relation* xIy.

The Five Conditions

An examination of Arrow's five conditions shows that they do not make sense in terms of the model in which R is primary and P and I are derived. We examine the conditions and rewrite them without reference to P and I. Later we take the opposite tack and make P and I primary and R derivative and examine the conditions again from this point of view. In this interpretation, changes are also required in the conditions.

Condition 1 requires a "free triple" of alternatives $^{i.e.}$ there are three alternatives among which there can be any possible combination of individual orderings. As stated it is OK.

Condition 2, the positive association of social and individual values, is as follows:

Let R_1, ..., R_n and R'_1, ..., R'_n be two sets of individual ordering relations, R and R' the corresponding social orderings, and P and P' the corresponding social preference relations. Suppose that for each i the two individual ordering relations are connected in the following ways: for x' and y' distinct from a given alternative x, $x'R_i'y'$ if and only if $x'R_iy'$; for all y', xR_iy' implies $xR_i'y'$; for all y', $xP_i y'$ implies $xP_i'y'$. Then, if xPy, xP'y.

Since P information is assumed not to be available in accordance with the above discussion, we must delete references to P and rewrite the condition only in terms of R as follows.

Condition 2': Let $R_1, ..., R_n$ and $R'_1, ..., R'_n$ be two sets of individual ordering relations, R and R' the corresponding social orderings. Suppose that for each i the two individual ordering relations are connected in the following ways: for x' and y' distinct from a given alternative x, $x'R_i'y'$ if and only if $x'R_iy'$; for all y', xR_iy' implies xR'_iy'; $\{xR_iy', y'R_ix\}$ implies $\{xR'_iy', y'R'_ix\}$ EOR $xR_i'y'$. Then, if xRy, xR'y.

However, Arrow's logical statement of the positive association of social and individual values is too narrow compared with his verbal statement: "...if one alternative social state rises or remains still in the ordering of every individual without any other change in those orderings, we expect that it rises, or at least does not fall, in the social ordering." An example will suffice to point out Arrow's lack of consistency between his verbal and logical statements of positive association. Suppose in all the individual orderings alternative x rises or remains the same. The old orderings are denoted R and the new orderings, R'. Let the old social ordering be aRbRxRcRdRe, for example, and the new ordering be cR'xR'bR'eR'aR'd. Then x has definitely risen in the social ordering since it has gone from third place to second place. However, when we break the old and new social choices down into their binary constituents, we have the following: aRx, bRx, xRc, xRd, xRe and cR'x, xR'b, xR'e, xR'a, xR'd. Even though x has risen in the social ordering, xRc and cR'x in violation of Arrow's statement of condition 2. Therefore, condition 2, as logically stated, is too restrictive. Arrow requires that x be preferred or indifferent to *every* alternative in the new ordering that it is preferred or indifferent to in the old ordering while his verbal statement only requires that x rise in the *social ordering.* Also Arrow requires that all the other alternatives besides x maintain their same places in the social orderings between old and new. In the above example, for instance, dRe and eR'd. Whether e and d rise or fall with respect to each other in the social ordering has nothing to do with whether x rises or falls in the social ordering given that x rises or remains the same for each individual. Arrow is restricting the SWF unnecessarily by requiring that "if xPy, then xP'y" and also that "for x' and y' distinct from a given alternative x, $x'R_i'y'$ if and only if $x'R_iy'$". Therefore, his proof does not apply to the more general case which is indicated by his verbal statement of the condition.

In our restatement of the condition we will not require relationships among alternatives other than x to remain constant in the new ordering nor will we require that if xRy', then xR'y' for any specific y'. We do require that if x has a certain rank in the old ordering, it will have that rank or higher in the new ordering. Rank is defined as the difference in the number of alternatives to which x is preferred or indifferent and the number of alternatives that are preferred or indifferent to x and can be positive or negative. If xRy for s values of y in the old environment, then xR'y for $r \geq s$ values of y in the new environment in order for the rank to increase or remain the same.

We, therefore, have a new restatement of Condition 2:

Condition 2'': Let $R_1, ..., R_n$ and $R'_1, ..., R'_n$ be two sets of individual ordering relations, R and R' the corresponding social orderings. Suppose that for each i the two individual ordering relations are connected in the following ways: for all y', xR_iy' implies xR'_iy'; $\{xR_iy', y'R_ix\}$ implies $\{xR'_iy', y'R'_ix\}$ EOR $xR_i'y'$. Then the rank of x in the social ordering R' is increased over its rank in R or remains the same where rank is defined as the difference in the number of alternatives to which x is preferred and the number of alternatives which are preferred to x.

When ties are considered, since there is in general no one to one correspondence between the elements of the tie in the old and new orderings, we can only require that the average rank of x over all members of the tie solution should increase or remain the same. An example is R = {xRyRz, xRzRy, yRxRz, yRzRx}. R' = {zR'x R'y, xR'zR'y, yR'zR'x}. The average rank of z has increased between R and R'.

Condition 3:

"Let R_1, ..., R_n and R'_1, ..., R'_n be two sets of individual orderings and let C(S) and C'(S) be the corresponding social choice functions. If, for all individuals i and all x and y in a given environment S, xR_iy if and only if $xR_i'y$, then C(S) and C'(S) are the same."

Since this Condition doesn't involve the explicit use of P, it is acceptable under the assumption that R is primary and P and I, derivative except for the following observations. Arrow clearly intends for S to include fewer alternatives than there are in the set corresponding to R_1, ..., R_n and R'_1, ..., R'_n. Therefore, C(S) and C'(S) are not the social choice functions corresponding to R_1, ..., R_n and R'_1, ..., R'_n. Let T be the set corresponding to R_1, ..., R_n and R'_1, ..., R'_n. Then Condition 3 can be restated as follows:

Condition 3': Let R_1, ..., R_n and R'_1, ..., R'_n be two sets of individual orderings over a set T. If, for all individuals i and all x and y in a given environment $S \subset T$, xR_iy if and only if $xR_i'y$, then C(S) and C'(S) are the same.

Arrow clearly intends for the individual data to be the same for the two sets of individual orderings over the set S. Therefore, since there are three possibilities, xR_iy, yR_ix and $\{xR_iy, yR_ix\}$, there must be a "if and only if" statement for yR_ix or else there could be some individual switching between yR_ix and $\{xR_iy, yR_ix\}$. Therefore, Condition 3 must be changed to the following:

Condition 3'': Let R_1, ..., R_n and R'_1, ..., R'_n be two sets of individual orderings over a set T. If, for all individuals i and all x and y in a given environment $S \subset T$, xR_iy if and only if $xR_i'y$ and yR_ix if and only if $yR_i'x$, then C(S) and C'(S) are the same.

It's worth taking another look at this condition since it is used to justify the decomposition of the social ordering among m alternatives into binary orderings which are then assumed to be the same as the social orderings when only two alternatives are considered at a time. Arrow makes no distinction among values of R as a function of m, the number of alternatives. He states: “If, then, we know C([x,y]) for all two-element sets, we have completely defined the relations P and I and therefore the relation R; but, by Definition 3, knowing the relation R completely determines the choice function C(S) for all sets of alternatives. Hence, one of the consequences of the assumptions of rational choice is that the choice *in any environment* can be determined by a knowledge of the choices in two-element environments.” (italics added) Arrow is confusing here the decomposition of R which is a function of m into binary components which can always be done by virtue of transitivity, and the binary social orderings made for all two-element sets. They are not necessarily the same. In fact, as we shall see, Condition 3 provides for binary independence but does not imply that the binary decomposition of the social ordering over the set T containing m alternatives is equal to the set of binary social orderings. We write R^m to emphasize the dependence of the social ordering on the number of alternatives being ordered.

For example, let’s consider the social ordering $aR^5bR^5cR^5dR^5e$ which, by virtue of transitivity, can be decomposed into the following set of binary social orderings: {aR^5b, aR^5c, aR^5d, aR^5e, bR^5c, bR^5d, bR^5e, cR^5d, cR^5e, dR^5e}. In general, these are not the same as the social orderings among alternatives taken two at a time which, for example, might be {bR^2a, aR^2c, aR^2d, eR^2a, bR^2c, dR^2b, bR^2e, cR^2d, eR^2c, dR^2e}. It is not even required by Condition 3 that the set of orderings formed by the decomposition of the m-ary ordering into binary constituents and selecting just those orderings involving alternatives in S be the same as the set of binary social orderings involving alternatives in S—only that the binary constituents be independent of other individual ordering information.

From Arrow’s definition of a SWF we have $f(R_1, ..., R_n) = R$ where f is the SWF. In general this is not the same as the recomposition of the functions $f_{xy}(R_1, ..., R_n)$ for all values of the alternatives x and y where f_{xy} is the binary SWF, f is the m-ary SWF and there are m alternatives altogether. Therefore, C(S) is not, in general,
$C(\{f_{xy}(R_1, ..., R_n)\}|$all values of x and y in S). In words, the social choice over the set S = {x_1,

$x_2,...x_m$) in which the elements of S are related by an m-ary relationship, $x_1R^m\ x_2R^m...\ R^mx_m$ is not the same as the social choice over the set S = {x_1, $x_2,...x_m$) in which the elements of S are related by binary relationships: $x_1R^2\ x_2$, $x_1R^2\ x_3$, ..., $x_2R^2\ x_3$, $x_2R^2\ x_4$, ..., $x_{m-1}R^2\ x_m$. The reason is that it is not required by Condition 3.

In Arrow's Condition 3 the social choices C(S) and C'(S) are related to the social orderings R and R' which are in turn related to the individual orderings R_1, ..., R_n and R'_1, ..., R'_n. How the orderings over the set S are related to the orderings R and R' is never specified. Arrow does give some verbal justification for Condition 3 as follows: "Suppose an election is held, with a certain number of candidates in the field, each individual filing his list of preferences, and then one of the candidates dies. Surely the social choice should be made by taking each of the individuals' preference lists, blotting out completely the dead candidate's name, and considering only the orderings of the remaining names in going through the procedure of determining a winner. That is, the choice to be made among the set S of surviving candidates should be independent of the preferences of individuals not in S. To assume otherwise would be to make the result of the election dependent on the obviously accidental circumstance of whether a candidate died before or after the date of polling."

None of this requires that, if the candidate who dies is y and if the social choice is x before the candidate dies, that it must be x after the candidate dies, only that it be based on individual orderings that do include y in the first case and don't in the second. It is not required that the social choice be the same as alternatives are added or deleted—only that those alternatives be added or deleted from each individual's list before the SWF produces the social ordering from which the social choice is obtained. If Arrow intended that the social choice be the same in every case regardless of the number of alternatives, this should have been stated.

Arrow goes on to say: "Alternatively stated, if we consider two sets of individual orderings such that, for each individual, his ordering of those particular alternatives in a given environment is the same each time, then we require that the choice made by society from that environment be the same when individual values are given by the first set of orderings as they are when given by the second."

The problem is that Arrow's words indicate that the choice in both cases be made from the "given environment," the set S. Obviously, the choice will be the same. Therefore, Arrow must be referring to the environment created by the decomposition of the R^m social ordering into binary constituents. Clearly, the social choice function could operate over that subset of the binary constituents containing alternatives belonging to the set S. However, this set of binary constituents is not necessarily equal to the set of binary constituents produced by the SWF operating on individual data containing only alternatives belonging to S. Nowhere in Condition 3 is any mention made that the binary social ordering between x and y be invariant regardless of the number of alternatives. It is not required either by Arrow's verbal or formal statement of Condition 3.

Binary independence and the binary decomposition of R^m being a function of the ordering produced by the binary SWF are not the same thing. Let f^m be the m^{th} stage SWF. Then if the binary decomposition of R^m is a function of the binary social ordering, we have:

For all x and y: $R^m|^{x,y} = h\{f^2([R_i|^{xy}]\}$ where f^m = SWF for m alternatives and $R|^{x,y}$ is the binary decomposition of R over x and y.

In general, $f^2([R_i|^{xy}]$ need not be equal to $R^m|^{x,y}$ in order for binary independence to hold. Binary independence will hold if $R^m|^{x,y}$ is a function of $[R_i|^{xy}]$ $^{i.e.}$ $R^m|^{x,y} = f^m_{xy}[R_i|^{xy}]$. Note that Arrow's specification of Condition 3 does not require that $R^m|^{x,y}$ be a function of $f^2([R_i|^{xy}]$— only that $R^m|^{x,y} = R'^m|^{x,y}$.

Arrow's alternative verbal statement does not elucidate the situation and should be changed to reflect the fact that the relationships among the alternatives in S refered to in Condition 3 are *derived* from the relationships in the social orderings R and R' over T and are not specified by the SWF operating upon individual orderings over the set S. Condition 3 does specify binary independence in the sense that for any given stage, the social ordering between x and y is only a function of the individual data concerning x and y. However, that ordering can vary from stage to stage where the stage number represents the number of alternatives being considered. If Arrow intends that there be stage to stage *invariance,* then this should be so stated. Verbally, what is required is something like the following:

If we consider a set of individual orderings over the set T and another set over the set S $\subset$ T, then the social choice is the same in each case provided that the social choice is in the set S.

This could be strengthened as follows:

If we consider a set of individual orderings over the set T and another set over the set S $\subset$ T, then the set of binary decompositions of the social ordering over S is equal to the set of binary decompositions of the social ordering over T truncated to include only those alternatives in the set S.

One might also consider any function g which would transform R^m to $R^{m'}$ instead of requiring strict equality between binary constituents. We call this a "reduction" from R^m to $R^{m'}$ and g the reduction function. We would then have a modified Condition 3 as follows:

Condition 3''': Let $R_1, ..., R_n$ and $R'_1, ..., R'_n$ represent individual orderings over S and T, respectively, (S $\subset$ T, T contains m alternatives and S contains m' alternatives) with social orderings R^m and $R'^{m'}$. xR_iy if and only if $xR_i'y$ and yR_ix if and only if $yR_i'x$ for all x and y in S. Let there be a function g independent of m that reduces R^m as follows: $R^{m'} = g(R^m)$. Then $R^{m'}$ and $R'^{m'}$ are the same.

For tie solutions, Condition 3''' is all that is necessary to insure a rational relationship among social orderings over alternative sets S and T consisting of different numbers of alternatives where $S \subset T$. However, there is a special case corresponding to Arrow's specification for "blotting out" the dead candidates. That would be to blot out the dead candidates from each element of the tie solution and then combine terms, if necessary, to obtain a solution identical (rather than a function of) to the solution for the reduced number of alternatives.

Condition 4: The social welfare function is not to be imposed.

Definition 5: A social welfare function will be said to be imposed if, for some pair of distinct alternatives x and y, xRy for any set of individual orderings $R_1, ..., R_n$, where R is the social ordering corresponding to
$R_1, ..., R_n$.

Condition 4 needs no changes.

Condition 5: The social welfare function is not to be dictatorial.

Definition 6: A social welfare function is said to be dictatorial if there exists an individual i such that, for all x and y, xP_iy implies xPy regardless of the orderings $R_1, ..., R_n$ of all individuals other than i, where P is the social preference relation corresponding to $R_1, ..., R_n$.

Changing P to R in Definition 6 makes it acceptable under the current assumptions.

Definition 6': A social welfare function is said to be dictatorial if there exists an individual i such that, for all x and y, xR_iy implies xRy regardless of the orderings $R_1, ..., R_n$ of all individuals other than i, where R is the social preference relation corresponding to $R_1, ..., R_n$.

P and I Primary

Although Arrow clearly specifies the possibility of ties in his model as stated, it is not his intention to include ties among orderings but only ties among alternatives via the indifference operator I. In addition, despite the fact that he specifies R first and derives P and I from R and also specifies his definition of a SWF in terms of R, his real primary values, as the subsequent development of his theory shows, are P and I and his intention is that individual and social data is specified in terms of P and I, not R. The model under these assumptions is really the following:

Axiom 1: For all x and y, one and only one of the following must be true: xPy, yPx or xIy.

Axiom 2:

1) For all x, y and z, if xPy and yPz, then xPz;
2) For all x, y and z, if xPy and yIz, then xPz;
3) For all x, y and z, if xIy and yPz, then xPz;
4) For all x, y and z, if xIy and yIz, then xIz.

Definition 1: xRy is defined to mean xPy EOR xIy.
where EOR is the exclusive or.

Now in this model ties can be included or excluded. It is clear that Arrow's intention is that they be excluded except for ties among alternatives. Since he doesn't take ties among orderings into account, his theory is essentially correct although an examination of his 5 Conditions shows some minor errors.

The Five Conditions in the P and I Model

Condition 1: Among all the alternatives there is a set S of three alternatives such that, for any set of individual orderings $T_1, ..., T_n$ of the alternatives in S, there is an admissible set of individual orderings $R_1, ..., R_n$ of all the alternatives such that, for each individual i, xR_iy if and only if xT_iy for x and y in S.

Analysis: The orderings R_i must be replaced by another symbol such as Q_i, since R in Condition 1 is clearly a stand-in for P or I and not the logical EORing of P and I. For T_i we can have the following: xP_iy, yP_ix, or xI_iy. Clearly, if xP_iy, in the T orderings, we must have xP_iy in the R orderings. But this is not necessarily the case as it is stated in Condition 1 since the condition xR_iy if and only if xT_iy will hold if xR_iy is xP_iy and xT_iy is xI_iy since $xR_iy \equiv xP_iy$ EOR xI_iy and $xT_iy \equiv xP_iy$ EOR xI_iy. Therefore, the condition as stated is incorrect. The correct statement would be the following:

Condition 1': Among all the alternatives there is a set S of three alternatives such that, for any set of individual orderings $T_1, ..., T_n$ of the alternatives in S, there is an admissible set of individual orderings $Q_1, ..., Q_n$ (where Q is a stand-in for P or I) of all the alternatives such that, for each individual i, xP_iy if and only if xP'_iy for x and y in S and xI_iy if and only if xI'_iy for x and y in S, where P_i and I_i represent orderings in Q and P'_i and I'_i represent orderings in T.

Condition 2: Let $R_1, ..., R_n$ and $R'_1, ..., R'_n$ be two sets of individual ordering relations, R and R' the corresponding social orderings, and P and P' the corresponding social preference relations. Suppose that for each i the two individual ordering relations are connected in the following ways: for x' and y' distinct from a given alternative x, $x'R_i'y'$ if and only if $x'R_iy'$; for all y', xR_iy' implies $xR_i'y'$; for all y', $xP_i y'$ implies $xP_i'y'$. Then, if xPy, xP'y.

Analysis: Again substitute Q_i for R_i in the individual orderings where Q is a stand-in for P or I. Arrow doesn’t cover the case, xIy. If x rises or remains the same in the judgment of each individual, then in the new state x would either rise to a preference or remain as an indifference. If xIy, then xR'y.

"$x'R_i'y'$ if and only if $x'R_iy'$" can be satisfied in any of the following ways: $x'P_i'y'$ and $x'P_iy'$, $x'P_i'y'$ and $x'I_iy'$, $x'I_i'y'$ and $x'P_iy'$, $x'I_i'y'$ and $x'I_iy'$, $y'P_i'x'$ and $y'P_ix'$, $y'P_i'x'$ and $y'I_ix'$, $y'I_i'x'$ and $y'P_ix'$. Clearly, Arrow intends for the exact relationship to hold in each of the two cases i'e' $x'P_i'y'$ if and only if $x'P_iy'$ and $x'I_i'y'$ if and only if $x'I_iy'$.

Arrow states: "The condition that x be not lower on the R_i' scale than x was on the R_i scale means that x is preferred on the R_i' scale to any alternative to which it was preferred on the old (R_i) scale and also that x is preferred or indifferent to any alternative to which it was formerly indifferent. The two conditions of the last sentence, taken together, are equivalent to the following two conditions: (1) x is preferred on the new scale to any alternative to which it was formerly preferred; (2) x is preferred or indifferent on the new scale to any alternative to which it was formerly preferred or indifferent." The second condition causes problems as follows: "for all y', xR_iy' implies $xR_i'y'$" can be satisfied by xP_iy' and $xP_i'y'$, xP_iy' and $xI_i'y'$, xI_iy' and $xP_i'y'$ or xI_iy' and $xI_i'y'$. Clearly, Arrow doesn't intend for this to be the case. Also, clearly, his statements (1) "x is preferred or indifferent to any alternative to which it was formerly indifferent" and (2) "x is preferred or indifferent on the new scale to any alternative to which it was formerly preferred or indifferent" are incompatible. Clearly, Arrow intends for (1) in the last sentence to be implemented. This can be implemented by the following statement: for all y', $xI_i y'$ implies $xR_i'y'$.

The truth table is as follows:

	$xP_i'y'$	$xI_i'y'$	$y'P_i'x$
xP_iy'	1	0	0
xI_iy'	1	1	0
$y'P_ix$	1	1	1

Condition 2 can be restated as follows:

Condition 2': Let $Q_1, ..., Q_n$ and $Q'_1, ..., Q'_n$ be two sets of individual ordering relations, Q and Q' the corresponding social orderings, and P and P' the corresponding social preference relations. Suppose that for each i the two individual ordering relations are connected in the following ways: for x' and y' distinct from a given alternative x, $x'P_i'y'$ if and only if $x'P_iy'$ and $x'I_i'y'$ if and only if $x'I_iy'$; for all y', $xP_i y'$ implies $xP_i'y'$; for all y', xI_iy' implies $xR_i'y'$;. Then, if xPy, xP'y and if xIy, xR'y.

When ties are considered, since we may not be dealing with the same set of ties as x rises, Condition 2' should hold on average over the set of ties.

However, Arrow's analysis of the Positive Association of Social and Individual values is too narrow. We take at face value his statement: "...if one alternative social state rises or remains still in the ordering of every individual without any other change in those orderings, we expect that it rises, or at least does not fall, in the social ordering." This general statement of positive association, however, does not translate exactly into Condition 2. An example will suffice to point out Arrow's lack of generality. Suppose in all the individual orderings alternative x rises or remains the same. The old orderings are denoted R and the new orderings, R'. Let the old social ordering be aRbRxRcRdRe, for example, and the new ordering be cR'xR'bR'eR'aR'd. Then x has definitely risen in the social ordering since it has gone from third place to second place. However, when we break the old and new social choices down into their binary constituents, we have the following: aRx, bRx, xRc, xRd, xRe and cR'x, xR'b, x R'e, x R'a, xR'd. Even though x has risen in the social ordering, xRc and cR'x in violation of Arrow's statement of Condition 2 which is too restrictive.

Arrow requires that x be preferred to *every* alternative in the new ordering that it is preferred to in the old ordering and preferred or indifferent in the new ordering to every alternative to which it is indifferent in the old ordering while his verbal statement only requires that x rise in the *social ordering.* Note that Arrow doesn't require that all the other alternatives besides x maintain their same places in the social orderings between old and new, however. In the above example, for instance, dRe and eR'd which is in accordance with Arrow's statement of Condition 2. Whether e and d rise or fall with respect to each other has nothing to do with whether x rises or falls in the social ordering given that x rises or remains the same for each individual. Arrow is restricting the SWF unnecessarily by requiring that if xPy, then xP'y, and therefore, his proof does not apply to the more general case which is indicated by his verbal statement of the condition. There does not have to be an exact correspondence between the binary relationships of the old and the new orderings although the old and new orderings can each be decomposed into binary relationships by virtue of transitivity.

In our restatement of the condition we will not require relationships among alternatives other than x to remain constant in the new ordering nor will we require that if xRy', then xR'y' for any specific y'. We do require that if x has a certain rank in the old ordering, it will have that rank or higher in the new ordering. Rank is defined as the difference in the number of alternatives to which x is preferred and the number of alternatives that are preferred to x and can be positive or negative. Considering only preferences for the moment, if xPy for s values of y in the old environment, then xP'y for $r \geq s$ values of y in the new environment in order for the rank to increase or remain the same. Considering preferences and indifferences, let S be the set of alternatives such that xPs, $s \in S$ and T be the set of alternatives such that tPx, $t \in T$. Then x will stay the same or increase in rank if O(S) - O(T) stays the same or increases where O(U) stands for the order of the set U. This takes into account that x may be indifferent to a set of alternatives which may increase or decrease in the new environment as compared with the old. The difference between the number of alternatives to which x is preferred and the number of alternatives which are preferred to it must remain the same or increase in the new environment in order for the rank of x to remain the same or increase. If there are alternatives in the old environment to which x is indifferent, and if the number of these alternatives increases or decreases in the new environment, then this increase or decrease must be such that the rank of x in the new environment does not decrease. At least as many of the increase in the indifference set must come from the set of alternatives that are preferred to x as from the set of alternatives to which x is preferred. Likewise, at least as many of the decrease in the indifference set must go to the set of alternatives to which x is preferred as go to the set of alternatives which are preferred to x.

Taking these restrictions away we have a new statement of Condition 2:

Condition 2'': Let $Q_1, ..., Q_n$ and $Q'_1, ..., Q'_n$ be two sets of individual ordering relations, Q and Q' the corresponding social orderings (where Q is a stand-in for P or I), P and P' the corresponding social preference relations and I and I' the corresponding social indifference relationships. Suppose that for each i the two individual ordering relations are connected in the following ways: for all y', $xP_i y'$ implies $xP_i'y'$; for all y', xI_iy' implies $xR_i'y'$. Then the rank of x in the social ordering R' will increase over its rank in R or remain the same where rank is defined as the difference in the number of alternatives to which x is preferred and the number of alternatives which are preferred to x.

When ties are considered, the average rank of x over the set of ties should increase or remain the same.

Condition 3:

Let $R_1, ..., R_n$ and $R'_1, ..., R'_n$ be two sets of individual orderings and let C(S) and C'(S) be the corresponding social choice functions. If, for all individuals i and all x and y in a given environment S, xR_iy if and only if $xR_i'y$, then C(S) and C'(S) are the same.

Again, $R_1, ..., R_n$ and $R'_1, ..., R'_n$ must be replaced by $Q_1, ..., Q_n$ and $Q'_1, ..., Q'_n$ Aside from the fact that "xR_iy if and only if $xR_i'y$" should be stated "xP_iy if and only if $xP_i'y$ and xI_iy if and only if $xI_i'y$" the major problem with this condition is that there must be a set of ordering relations over the set S, but Arrow doesn't say what these ordering relations are or how they are to be derived from R and R'. Obviously, these orderings cannot be produced by just considering the individual orderings over the set S and then applying the SWF to them because this would be tautological. Obviously, the SWF must be applied to the full set of orderings for R and R', and then the orderings over the set S derived from the orderings over R and R'. In other words the orderings over the set S are a function of the orderings over the full set, let's call it T. It's clear that this function must be the same for both R and R' so that $g(R_T) = R_S$ and $f(R'_T) = R'_S$ and that R_S and R'_S should be the same although Arrow only requires that C(S) and C'(S) be the same. Note, however, that R_S and R'_S need not be the same as the ordering produced by the SWF applied to the individual orderings for the set T. That is not required by Condition 3. C(S) is the "top slot" of the ordering R_S and C'(S) is the "top slot" of the ordering R'_S. Clearly, Arrow intends for us to take each of the orderings R and R' and "blot out" the alternatives not in S with the resultant ordering being the ordering over the set S. But Arrow never specifies this function mathematically. A specification of the function, g, is the following:

Let R_T be an ordering over the set T [O(T) = t] and $R_{S|T}$ be a reduction of R_T which is an ordering over the set S [O(S) = s], $S \subset T$.

$R_T = a_1R^1a_2R^2...a_iR^i...a_{t-1}R^{t-1}a_t$ and

$R_{S|T} = b_1R^1b_2R^2...b_jR^j...b_{s-1}R^{s-1}b_s$. g is given by the following algorithm:

$R_T^0 = 1$; $R^t = 1$; j=0.

For i=1 to i=t, if $a_i \in S$, then $R_T^{j+1} = R_T^j\, a_iR^i$; j=j+1, $R^j = R^i$. If $a_i \notin S$,

if R^j =I AND R^i = P, $R_T^j = R_T^j(P/I)$, R^j=P

As an example, let R_T = aPbIcIdIePf and R'_T = aIbPdIcIePf and S = {a,c,e,f}.

R_T^0 =1; R^6=1; j=0

i=1, $a_1 \in S$, $R_T^1 = 1 \cdot aP$, j=1, R^1=P

i=2, $a_2 \notin S$, R^1=P, R^2=I

i=3, $a_3 \in S$, R_T^2=aPcI, j=2, R^2=I

i=4, $a_4 \notin S$, R^2=I, R^3=I

i=5, $a_5 \in S$, R_T^3=aPcIeP, j=3, R^3=P

i=6, $a_6 \in S$, R_T^4=aPcIePf $\cdot$1

$R_{S|T}$=aPcIePf

$R'_T{}^0$=1; R^6=1; j=0.

i=1, $a_1 \in S$, $R_T{}^1$= 1·aI, j=1, R^1=I

i=2, $a_2 \notin S$, R^1=I, R^2=P, $R_T{}^1$=aP, R^1=P

i=3, $a_3 \notin S$, R^1=P, R^3=I

i=4, $a_4 \in S$, $R_T{}^2$ = aPcI, j=2, R^2 = I

i=5, $a_5 \in S$, $R_T{}^3$ = aPcIeP, j=3, R^3=P

i=6, $a_6 \in S$, $R_T{}^4$ = aPcIePf ·1

$R'_{S|T}$=aPcIePf = $R_{S|T}$

The function g will work so long as R_T and R'_T have the set S in the same order. But there is no requirement that they be in the same order in R_T as in R'_T, only that they be transformed into the same order by whatever function converts R_T and R'_T into the reduced form of R_T and R'_T, $R_{S|T}$ and $R'_{S|T}$. Therefore, the reduction function, as we shall call it, is related to the SWF and is not necessarily the same for every SWF.

When ties are considered, Condition 3 need not be changed at all. The reduction function must convert the solutions R_T and R'_T where one or both of R_T and R'_T consist of ties to $R_{S|T}$ and $R'_{S|T}$ where $R_{S|T}$ and $R'_{S|T}$ consist of the same identical orderings whether ties or singular solutions.

Condition 4: The social welfare function is not to be imposed.

A social welfare function will be said to be imposed if, for some pair of distinct alternatives x and y, xRy for any set of individual orderings R_1, ..., R_n, where R is the social ordering corresponding to R_1, ..., R_n.

This means that, no matter what the individual orderings, society can never choose yPx. But it can choose xPy or xIy. This seems to be an ambiguous imposition. If it were truly imposed then the choice would be xPy regardless of individual orderings or xIy regardless of individual orderings, but the way the condition is stated, individual orderings can decide between xPy and xIy. A better definition of imposition would be the following:

Definition 5': A social welfare function will be said to be imposed if, for some pair of distinct alternatives x and y, xPy for any set of individual orderings $Q_1, ..., Q_n$, or xIy for any set of individual orderings
$Q_1, ..., Q_n$ where Q is a stand-in for P or I.

Condition 5: The social welfare function is not to be dictatorial.

Definition 6: A social welfare function is said to be dictatorial if there exists an individual i such that, for all x and y, xP_iy implies xPy regardless of the orderings $R_1, ..., R_n$ of all individuals other than i, where P is the social preference relation corresponding to $R_1, ..., R_n$.

In addition to changing the R's to Q's, the only change to be made here in Definition 5 would be to include the case of the dictator being indifferent between x and y as follows:

Definition 6': A social welfare function is said to be dictatorial if there exists an individual i such that, for all x and y, xP_iy implies xPy and xI_iy implies xIy regardless of the orderings $Q_1, ..., Q_n$ of all individuals other than i, where P is the social preference relation and I is the social indifference relation corresponding to $Q_1, ..., Q_n$.

The Implications of Inclusion of Ties for Both Models

For the first model in which R is primary and P and I derivative, Arrow explicitly states that ties are possible. This is the model that Arrow puts forth although his subsequent development assumes the second model in which P and I are primary and R is either derivative or a stand-in for P or I. For the R primary model Arrow states that ties are possible. However, he means that only ties between alternatives and not ties between orderings are to be considered. He defines ties between alternatives as indifferences. Since he does mention ties, the door is opened for a more general consideration of ties in the R primary model.

We now proceed to demonstrate solutions which are social orderings for a specific SWF for the case m = 3 for the R primary model. Much of this follows Lawrence, 1998. Let us assume alternatives x, y and z and n (odd) voter/consumers. We will assume that "knowing the social choices made in pairwise comparisons determines the entire social ordering," although, as we've seen, this is not guaranteed by Condition 3. Accordingly, we consider the social choices of the alternatives two by two. Our SWF is as follows. If N(x,y) > N(y,x), then xRy. If N(y,x) > N(x,y), then yRx. At the ternary level we have 8 cases:

Case 1:	xRy, xRz, yRz
Case 2:	xRy, xRz, zRy
Case 3:	xRy, zRx, yRz
Case 4:	xRy, zRx, zRy
Case 5:	yRx, xRz, yRz
Case 6:	yRx, xRz, zRy
Case 7:	yRx, zRx, yRz
Case 8:	yRx, zRx, zRy

According to the Condorcet (1785) method for determining the outcome of an election, we consider each of the alternatives in pairs, determine the winner for each pair and then determine the final social ordering by combining these results. We use the Condorcet method in our SWF for the above cases in which it actually produces a result. Therefore, we have the following:

Case	Social Ordering
1	xRyRz
2	xRzRy
4	zRxRy
5	yRxRz

7	yRzRx
8	zRyRx

This leaves only cases 3 and 6. Consider the solution {xRyRz, yRzRx, zRxRy} for Case 3. We call a reduced ordering or reduced solution an ordering with one or more alternatives removed. If we consider {xRyRz, yRzRx, zRxRy} and remove z, we get {xRy, yRx, xRy}. Combining terms we have {2xRy, yRx}. If we choose the most numerous of xRy and yRx as the solution, we get xRy by 2 to 1 which we know to be true.

Likewise, if we reduce {xRyRz, yRzRx, zRxRy} by y, we get {xRz, zRx, zRx} or {xRz, 2zRx}. 2zRx > xRz and we take zRx as the reduced solution which agrees with the known binary solution. Similarly, if we remove x from the social solution, we have {yRz, yRz, zRy} which yields yRz. Accordingly, our SWF algorithm is as follows:

1) Choose the Condorcet solution if it exists.
2) If the Condorcet solution doesn't exist, construct a solution such that, when the solution is reduced by any single alternative, the most numerous of the remaining binary relationships is the same as the known binary solution.

Notice that our algorithm will always produce consistent results if the ternary solution is generated from the binary solution in such a way that there is a 2 to 1 ratio between the correct binary solution and the incorrect binary solution and then we take the larger of the two as our reduced solution. We construct our solutions in this manner in order to be compliant with Arrow's Condition 3. Satisfying the other Conditions is then trivial as can be shown. Whether or not such a solution always exists will be answered affirmatively elsewhere (Lawrence, 1998). Here all we need to show is the existence of a solution for Case 6. Consider the solution {yRxRz, xRzRy, zRyRx}. Reduction by z yields yRx; by y, xRz; by x, zRy which agrees with the known binary case and is consistent with the above definition.

Therefore, we have demonstrated a consistent algorithm for the SWF which yields the same social orderings when reduced from the ternary case to the binary case as those produced at the binary level directly from the domain. There is complete consistency of social orderings and not just of alternatives produced by the choice function. The choice function only produces the top position in an ordering. We demand consistency over all orderings which can be produced by reducing a social ordering and this strengthens Arrow's Condition 3.

The Ternary Case — n even

When n is even we have a total of 27 cases. We have already considered the first 8 cases above. For convenience we define {xRy, yRx} as xTy. In addition there is one more tie possibility, a three way tie:

N(x,y) = N(y,x) = N(y,z) = N(z,y) = N(x,z) = N(z,x). We write this as {xRy, yRx, yRz, zRy, xRz, zRx} and define this as xTyTz. Solutions for the remaining cases are shown below.

Case	Binary Solutions	Ternary Solution
9:	xRy, xRz, yTz	xRyTz
10:	xRy, zRx, yTz	{zRxRy, xRyTz, yTzRx}
11:	yRx, xRz, yTz	{yRxRz, xRyTz, yTzRx}
12:	yRx, zRx, yTz	yTzRx
13:	xRy, xTz, yRz	{xRyRz, yRxTz, xTzRy}
14:	xRy, xTz, zRy	xTzRy
15:	yRx, xTz, yRz	yRxTz
16:	yRx, xTz, zRy	{zRyRx, yRxTz, xTzRy}
17:	xTy, xRz, yRz	xTyRz
18:	xTy, xRz, zRy	{xRzRy, xTyRz, zRxTy}
19:	xTy, zRx, yRz	{yRzRx, xTyRz, zRxTy}
20:	xTy, zRx, zRy	zRxTy
21:	xRy, xTz, yTz	{xRyTz, xTzRy, xTyTz}
22:	yRx, xTz, yTz	{yRxTz, yTzRx, xTyTz}
23:	xTy, xRz, yTz	{xRyTz, xTyRz, xTyTz}
24:	xTy, zRx, yTz	{zRxTy, yTzRx, xTyTz}
25:	xTy, xTz, yRz	{yRxTz, xTyRz, xTyTz}
26:	xTy, xTz, zRy	{zRxTy, xTzRy, xTyTz}
27:	xTy, xTz, yTz	xTyTz

P and I Primary Model

In the P and I primary model, from Arrow's point of view, there is no need to mention ties since they are covered by the I operator although again the I operator only provides for ties between alternatives and not for ties among orderings. If this model is assumed, then Arrow's Impossibility Theorem may be true. In general xIy is not the same as {xPy, yPx}. xIy means that an individual or society considers the alternatives x and y to be indistinguishable. {xPy, yPx} means that an individual or society is divided between xPy and yPx. The consideration of ties in this model would include the following possibilities: {xPy, yPx}, {xPy, xIy}, {yPx, xIy}, {xPy, yPx, xIy}.

We now proceed to develop a set of solutions for the P and I primary model for m=3. In the P and I primary model, we have two relationships to deal with. Let N(x,y) be the number of individual voter/consumers who prefer x to y, and M(x,y) be the number who are indifferent between x and y. There are then 13 possibilities as follows:

Case 1 :	N(x,y) > N(y,x) > M(x,y)
Case 2:	N(y,x) > N(x,y) > M(x,y)
Case 3:	N(x,y) > M(x,y) > N(y,x)
Case 4:	N(y,x) > M(x,y) > N(x,y)
Case 5:	M(x,y) > N(x,y) > N(y,x)
Case 6:	M(x,y) > N(y,x) > N(x,y)
Case 7:	N(x,y) > N(y,x) = M(x,y)
Case 8:	N(y,x) > N(x,y) = M(x,y)
Case 9:	M(x,y) > N(x,y) = N(y,x)
Case 10:	N(x,y) = N(y,x) > M(x,y)
Case 11:	M(x,y) = N(x,y) > N(y,x)
Case 12:	M(x,y) = N(y,x) > N(x,y)
Case 13:	M(x,y) = N(x,y) = N(y,x)

One possible binary decision rule might the following. If N(x,y) > N(y,x) and M(x,y), then xPy. If N(y,x) > N(x,y) and M(x,y), then yPx. If M(x,y) > N(x,y) and N(y,x), then xIy. If N(x,y) = N(y,x) > M(x,y), then {xPy, yPx}. If N(x,y) = M(x,y) > N(y,x), then {xPy, xIy}. If N(y,x) = M(x,y) > N(x,y), then
{yPx, xIy}. If N(x,y) = N(y,x) = M(x,y), then {xPy, yPx, xIy}. There would be 7 possible

social orderings at the binary level. At the ternary level would be 7^3 possible combinations each of which would require a social ordering.

However, the SWF need not make use of every possible range element in providing a mapping from domain to range. We only need to make sure that there is at least one set of connections which satisfy Arrow's criteria and axioms. Accordingly, we only consider the following binary social orderings: xPy, yPx, xTy = {xPy, yPx}, xIy, and the following binary decision rule.

		Social Ordering
Case 1 :	N(x,y) > N(y,x) > M(x,y)	xPy
Case 2:	N(y,x) > N(x,y) > M(x,y)	yPx
Case 3:	N(x,y) > M(x,y) > N(y,x)	xPy
Case 4:	N(y,x) > M(x,y) > N(x,y)	yPx
Case 5:	M(x,y) > N(x,y) > N(y,x)	xIy
Case 6:	M(x,y) > N(y,x) > N(x,y)	xIy
Case 7:	N(x,y) > N(y,x) = M(x,y)	xPy
Case 8:	N(y,x) > N(x,y) = M(x,y)	yPx
Case 9:	M(x,y) > N(x,y) = N(y,x)	xIy
Case 10:	N(x,y) = N(y,x) > M(x,y)	xTy
Case 11:	M(x,y) = N(x,y) > N(y,x)	xPy
Case 12:	M(x,y) = N(y,x) > N(x,y)	yPx
Case 13:	M(x,y) = N(x,y) = N(y,x)	xTy

At the ternary level we have $64 = 4^3$ cases to consider as follows. We present the solutions in Appendix 1.

THEOREM For m = 3 and any n there exists a SWF relative to the relations P and I for which the social orderings consist of either unique rankings or of ties of at most three orderings.

PROOF By inspection.

Conclusions

We have shown that Arrow's assertion that he included the consideration of ties in his model boils down to the inclusion of ties among alternatives and not ties among orderings. A closer examination reveals that, while a tie between two binary orderings, {xRy, yRx}, is acceptable in his model, the indifference operator, I, is defined as xRy AND yRx. Therefore, ties need not be considered further since they have been defined as indifferences. Heuristically, it doesn't make sense to define a social choice as an indifference, xIy, if half the individuals specify xP_iy and half specify yP_ix. In such a case society is not indifferent at all but evenly divided.

Arrow's model postulates the relationship R as primary, and then preference, P, and indifference, I, are defined in terms of R. His definition of a SWF is one in which individuals specify R orderings as their individual preference orderings and society chooses an R ordering as the social ordering. It is shown that for R primary and P and I derivative, Arrow's axioms, definitions and conditions don't make sense. Arrow really intends for individuals to specify orderings in terms of P and I and for society to specify its ordering in terms of P and I. This would necessitate a reexamination and reconsideration of the wording of these respective axioms, definitions and conditions. We conclude that Arrow did not word them correctly. A third interpretation of R is that it is used as a stand-in for P or I as opposed to being defined logically in terms of them. In this interpretation, for example, xRyRz, could be in fact xPyIz or xIyPz or xPyPz or xIyIz. This "stand-in" interpretation is also used by Arrow but is not consistent with his definition of R.

We examine a SWF for 3 alternatives when ties among orderings are allowed for the two cases: (1) R is primary and P and I are derivative; and (2) P and I are primary and R is derivative. We conclude that such a function exists for this special case which complies with the reformulated axioms, definitions and conditions. Elsewhere, it is shown that such a function exists for any number of alternatives and voters (Lawrence, 1998).

Appendix 1

Case	Binary Solutions	Ternary Solution
1	xPy, xPz, yPz	xPyPz
2	xPy, xPz, zPy	xPzPy
3	xPy, zPx, yPz	{xPyPz, yPzPx, zPxPy}
4	xPy, zPx, zPy	zPxPy
5	yPx, xPz, yPz	yPxPz
6	yPx, xPz, zPy	{yPxPz, xPzPy, zPyPx}
7	yPx, zPx, yPz	yPzPx
8	yPx, zPx, zPy	zPxPy
9	xPy, xPz, yIz	xPyIz
10	xPy, zPx, yIz	{zPxPy, xPyIz, yIzPx}
11	yPx, xPz, yIz	{yPxPz, xPyIz, yIzPx}
12	yPx, zPx, yIz	yIzPx
13	xPy, xIz, yPz	{xPyPz, yPxIz, xIzPy}
14	xPy, xIz, zPy	xIzPy
15	yPx, xIz, yPz	yPxIz
16	yPx, xIz, zPy	{zPyPx, yPxIz, xIzPy}
17	xIy, xPz, yPz	xIyPz
18	xIy, xPz, zPy	{xPzPy, xIyPz, zPxIy}
19	xIy, zPx, yPz	{yPzPx, xIyPz, zPxIy}
20	xIy, zPx, zPy	zPxIy
21	xPy, xIz, yIz	{xPyIz, xIzPy, xIyIz}
22	yPx, xIz, yIz	{yPxIz, yIzPx, xIyIz}
23	xIy, xPz, yIz	{xPyIz, xIyPz, xIyIz}
24	xIy, zPx, yIz	{zPxIy, yIzPx, xIyIz}

25	xIy, xIz, yPz	{yPxIz, xIyPz, xIyIz}
26	xIy, xIz, zPy	{zPxIy, xIzPy, xIyIz}
27	xIy, xIz, yIz	xIyIz
28	xPy, xPz, yTz	xPyTz
29	xPy, zPx, yTz	{zPxPy, xPyTz, yTzPx}
30	yPx, xPz, yTz	{yPxPz, xPyTz, yTzPx}
31	yPx, zPx, yTz	yTzPx
32	xPy, xTz, yPz	{xPyPz, yPxTz, xTzPy}
33	xPy, xTz, zPy	xTzPy
34	yPx, xTz, yPz	yPxTz
35	yPx, xTz, zPy	{zPyPx, yPxTz, xTzPy}
36	xTy, xPz, yPz	xTyPz
37	xTy, xPz, zPy	{xPzPy, xTyPz, zPxTy}
38	xTy, zPx, yPz	{yPzPx, xTyPz, zPxTy}
39	xTy, zPx, zPy	zPxTy
40	xPy, xTz, yTz	{xPyTz, xTzPy, xTyTz}
41	yPx, xTz, yTz	{yPxTz, yTzPx, xTyTz}
42	xTy, xPz, yTz	{xPyTz, xTyPz, xTyTz}
43	xTy, zPx, yTz	{zPxTy, yTzPx, xTyTz}
44	xTy, xTz, yPz	{yPxTz, xTyPz, xTyTz}
45	xTy, xTz, zPy	{zPxTy, xTzPy, xTyTz}
46	xTy, xTz, yTz	xTyTz
47	xPy, xIz, yTz	{xPyTz, yTzIx, zIxPy}
48	xPy, xTz, yIz	{xPyIz, yIzTx, zTxPy}
49	yPx, xIz, yTz	{yPxIz, xIzTy, zTyPx}
50	yPx, xTz, yIz	{yPxTz, xTzIy, zIyPx}
51	xIy, xPz, yTz	{xPzTy, zTyIx, yIxPz}

52	xIy, zPx, yTz	{zPxIy, xIyTz, yTzPx}
53	xTy, xPz, yIz	{xPzIy, zIyTx, yTxPz}
54	xTy, zPx, yIz	{zPxTy, xTyIz, yIzPx}
55	xIy, xTz, yPz	{yPzTx, zTxIy, xIyPz}
56	xIy, xTz, zPy	{zPyIx, yIxTz, xTzPy}
57	xTy, xIz, yPz	{yPzIx, zIxTy, xTyPz}
58	xTy, xIz, zPy	{zPyTx, yTxIz, xIzPy}
59	xIy, xIz, yTz	xIyTz
60	xIy, xTz, yIz	{xIyIz, yIzTx, zTxIy}
61	xTy, xIz, yIz	xTyIz
62	xIy, xTz, yTz	{xIyTz, yTzTx, zTxIy}
63	xTy, xIz, yTz	{xTyTz, yTzIx, zIxTy}
64	xTy, xTz, yIz	{xTyIz, yIzTx, zTxTy}

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