A New Approach to the Social Choice Function

by

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Abstract

In "Social Choice and Individual Values" Arrow (1951) discusses two kinds of functions: the Social Choice Function, SCF, and the Social Welfare Function, SWF. The SCF is a function whose range is an alternative or set of alternatives. The SWF is a function whose range is an ordering of alternatives or sets of alternatives. Arrow's famous General Possibility Theorem, GPT, states that no SWF exists which meets certain criteria. However, the SCF may well exist even in cases in which the SWF doesn't because the production of an alternative or set of alternatives is less restrictive than the production of a set of orderings over the entire alternative set.

The only role that the SCF plays in the GPT is in the specification of Condition 3, the Independence of Irrelevant Alternatives. In that condition Arrow specifies that the SCFs over a set $S \subset T$ (where T contains the total number of alternatives) produce the same social result in two different cases when the individual preference orderings within the set S are identical in both cases. The set T may have different orderings in the two cases, but the alternative or set of alternatives produced by the SCF must be the same. This is the weakest possible requirement for social orderings within the set S since it only requires the top alternative or set of alternatives to be the same. Stronger requirements would require that the top two or more alternatives within the set S be ordered in the same way, and the strongest requirement would be that the entire social orderings within the set S in the two cases be the same. The SWF produces orderings over T; a generalized SCF would produce orderings over S.

Arrow provides for tie solutions via the SCF but only for ties among alternatives. Since the SWF requires orderings and not alternatives, it is natural to examine ties among orderings and not just ties among alternatives. When ties among orderings are allowed (as demanded by the completeness axiom), Condition 3 can be strengthened to the maximum and solutions which violate the GPT can still be found. We demonstrate this for m, the number of alternatives, = 3 and 4.

Introduction

We assume a set, S, of m alternatives a,b,c ... and a relationship R which means "preferred or indifferent." Therefore, xRy means x is "preferred or indifferent" to y. For the present we will just consider binary orderings of this nature. Arrow (1951) sets down two axioms.

Axiom 1: "For all x and y, either xRy or yRx."

Axiom 2: "For all x, y, and z, xRy and yRz imply xRz."

Axiom 1 is the axiom of connection. Since xRx, R is reflexive, and since xRy or yRx for all alternatives in the set S, R is complete. Arrow also states (p. 13): "Note also that the word 'or' in the statement of Axiom 1 does not exclude the possibility of both xRy and yRx. That word merely asserts that at least one of the two events must occur; both may." Therefore we can restate Axiom 1 as follows:

For all x and y, one of the following must be true: 1) xRy; 2) yRx; 3) both xRy and yRx. We will use the nomenclature {xRy, yRx} for "both xRy and yRx."

Arrow (p. 14) then goes on to make two definitions:

Definition 1: "xPy is defined to mean not yRx."

Definition 2: "xIy means xRy and yRx."

where xPy is read "x is preferred to y" and xIy is read "x is indifferent to y."

A problem arises here in Arrow's logic. According to Axiom 1, one of the following must be true: 1) xRy; 2) yRx; 3) both xRy and yRx. If yRx is not true, then either (1) xRy or (3) both xRy and yRx must be true. Thus, Definition 1 doesn't make sense because xPy and (3) above cannot be true at the same time. Definition 1 conflicts with Axiom 1. Definition 2 is logically consistent but unnecessary. It also defines a tie as an indifference which limits the generality of the analysis. In addition, it stipulates a logical relationship between xRy and yRx: xIy $\equiv$ xRy AND yRx, where AND is the logical and, whereas Axiom 1 only requires that "both [events] may [occur]" which is heuristically a tie.

Arrow states: "Axioms 1 and 2 do not exclude the possibility that for some distinct x and y, both xRy and yRx. A strong ordering, on the other hand, is a ranking in which no ties are possible." Clearly, Arrow intends to define a tie as an indifference or he wouldn't imply that his analysis includes the possibility of a tie. The statement that "a strong ordering ... is a ranking in which no ties are possible" is also not true. Consider the strong ranking operator P. In an election between two candidates, x and y, in which half the voters prefer x to y and half prefer y to x, the result is clearly a tie between x and y. One can also consider the result to be a tie between the *orderings* xPy and yPx or using my notation: {xPy, yPx}.

Arrow further compounds the error of Definition 1 with

Lemma 1 (e): "For all x and y, either xRy or yPx."

It can not be too strongly emphasized that not xRy is yRx or {xRy, yRx}≡ xIy and not yPx. This makes sense heuristically since if y is not "preferred or indifferent to" x, then any of the following may be true: 1) x may be "preferred or indifferent" to y or 2) x may be indifferent to y or 3) x may be preferred to y. There is no way to know that xPy when the specification is xRy or yRx or xIy. If we could know that y is not preferred to x AND y is not indifferent to x, then we would know that xPy. But we only know that y is not "preferred or indifferent" to x. This is one of the problems of Arrow's analysis that leads to the erroneous conclusion that social choice is impossible.

The Social Choice Function

The choice function, C(S), is defined by Arrow (p. 15) as follows:

Definition 3: "C(S) is the set of all alternatives in S such that, for every y in S, sRy."

As such it can be used to specify ties among *alternatives* if the set, C(S), contains more than one element. Sen (1970, p. 48) says, "Arrow's impossibility theorem is precisely a result of demanding social orderings as opposed to choice functions." In other words, if the solutions required were simply alternatives, Arrow's Impossibility Theorem would not apply. Since Arrow only uses it in his specification of the Condition of the Independence of Irrelevant Alternatives, it would have been more natural (and certainly stronger) to define C(S) as an *ordering* over a subset instead of the highest ranking alternative or set of alternatives in a subset. Why, if orderings are required in the solution, should only top-ranking alternatives be required in one of the conditions? And why, if orderings are required, should ties only be allowed among alternatives and not among orderings? We define an ordering function herein which strengthens the Condition of Independence of Irrelevant Alternatives and allows for ties among orderings as well as ties among alternatives.

Arrow (p.15) makes the following incorrect statement: "Each element of C(S) is to be preferred to all elements of S not in C(S) and indifferent to all elements of C(S); and, therefore, if x belongs to C(S), xRy for all y in S." Assuming we have specified xRy, yRx or {xRy, yRx} for all x and y in S, we don't know whether one element is preferred to another element—only that one element is "preferred or indifferent" to another element. Therefore, we can't know that the elements of C(S) are "preferred to all elements of S not in C(S)," only that the elements of C(S) are "preferred or indifferent" to the elements not in C(S). For example, let's assume the following set S and the following relationships:

S = {x,y,z}

xRy, yRz, xRz

Then, clearly C(S) = x. y and z are not in C(S), but x is not preferred to y and z but it is "preferred or indifferent" to y and z and for every element, w, of S xRw which is all Definition 3 requires.

For x and y in C(S), xRy and yRx or {xRy, yRx}. For x in C(S) and z not in C(S), xRz and not zRx, not {xRz, zRx}. Arrow's definition of C(S) makes sense. However, it would be clearer if the following statement would be appended: For y in C(S), xRy and yRx. For y not in C(S), xRy.

Clearly, for x and y in C(S), xIy according to Definition 2 and this is the basis for Arrow's claim that he provides for ties. However, note that he has totally identified a tie as an indifference.

If the relationships among the alternatives in S are transitive according to Axiom 2 and either xRy or yRx but not both, then C(S) can contain only one element since the elements can be ordered, for example, as follows: xRyRz. If ties are possible, then, for example, we might have the following relationship:

{xRy, yRx}, yRz, xRz

Here, according to Arrow's Definition 2 we could write xIy and
C(S) = {x,y}. Then the elements of C(S) are indifferent to each other but are "preferred or indifferent" (not "preferred" as Arrow states) to the element, z, not in C(S).

If the relationships are intransitive, then C(S) might be empty. For example, if xRy, yRz and zRx, then C(S) is empty since there is no element that is "preferred or indifferent" to every other element.

Consider the example: {xRy, yRx}, {yRz, zRy}, xRz or xIy, yRz, xRz according to Definition 2. What is C(S)? Clearly, C(S) is {x,y} since xRy, xRz and yRx, yRz. This does not violate Arrow's Axiom 2 since xRy and yRz imply xRz. However, this does violate Arrow's Lemma 1(d): If xIy and yIz, then xIz, since xIy and yIz but xRz. Clearly, Arrow's definition of transitivity is incomplete because it doesn't provide for the tie case.

Arrow goes on to compound the confusion over xPy and not yRx. He states: “Conversely, suppose C[x,y] contains the single element x. Since y does not belong to C[x,y]), not yRx; by Definition 1, xPy.” But not yRx is xRy or {xRy, yRx} by Axiom 1, not xPy. The root of the problem is that xPy implies xRy but not vice versa. Not yRx does not imply xPy, but it implies either xRy or {xRy, yRx}. Similarly, xIy implies xRy, but xRy does not imply xIy.

Consider xRyRz. Arrow would have us believe that since not yRx, xPy and since not zRy, yPz and, therefore, xPz. However, if x is preferred or indifferent to y and y is preferred or indifferent to z, then there is the possibility that x is indifferent to y and y is indifferent to z and, therefore, x is indifferent to z.

Arrow’s Lemma 2 is also wrong.

LEMMA 2: A necessary and sufficient condition that xPy is that x be the sole element of C([x,y]).

However, if x is the sole element of C([x,y]), then xRy by Definition 3 and if xRy, C([x,y]) is x by definition. Therefore, xPy in Lemma 2 should be replaced by xRy.

Transitivity

So far we have been considering the relationship between just two alternatives: xRy. When there are three or more alternatives, then Arrow’s Axiom 2, the axiom of transitivity comes into effect. Transitivity limits the total number of logical possibilities to just those that heuristically and intuitively make sense. If x is “preferred or indifferent” to y and y is “preferred or indifferent” to z, then it makes sense that x should be “preferred or indifferent” to z. To say that z is “preferred or indifferent” to x in the last sentence would not make sense although xRy, yRz and zRx is a logical possibility. Thus transitivity imposes a rational limitation on the total number of logical possibilities. Another logical possibility, if xRy and zRy, then xRz or zRx, need not be stated since the implication does not imply a specific logical possibility. Either possibility is intuitively appropriate.

From Axiom 1 we know there are three possibilities: xRy, yRx or {xRy, yRx}. Arrow's Axiom 2, while correct insofar as it goes, does not take into account the tie case: {xRy, yRx}. Heuristically, Axiom 2 makes sense as long as there are no ties involved. Therefore, it is fair to ask how the tie case affects the axiom of transitivity. There are 3^3 logical combinations of xRy, yRx and {xRy, yRx}. There are only the following cases involving ties. The relationships which are marked transitive are those which make intuitive sense and are non-trivial in that only one of the two possible logical implications makes sense. The logic is Relationship 1 and Relationship 2 imply Relationship 3.

Case	Relationship 1	Relationship 2	Relationship 3	Transitive?
1	{xRy, yRx}	yRz	xRz	Yes
2	{xRy, yRx}	yRz	zRx	No
3	{xRy, yRx}	zRy	xRz	No
4	{xRy, yRx}	zRy	zRx	Yes
5	xRy	{yRz, zRy}	xRz	Yes
6	xRy	{yRz, zRy}	zRx	No
7	yRx	{yRz, zRy}	xRz	No
8	yRx	{yRz, zRy}	zRx	Yes
9	xRy	yRz	{xRz, zRx}	No
10	xRy	zRy	{xRz, zRx}	Trivial
11	yRx	yRz	{xRz, zRx}	Trivial
12	yRx	zRy	{xRz, zRx}	No
13	{xRy, yRx}	{yRz, zRy}	xRz	No
14	{xRy, yRx}	{yRz, zRy}	zRx	No
15	{xRy, yRx}	yRz	{xRz, zRx}	No
16	{xRy, yRx}	zRy	{xRz, zRx}	No
17	xRy	{yRz, zRy}	{xRz, zRx}	No
18	yRx	{yRz, zRy}	{xRz, zRx}	No
19	{xRy, yRx}	{yRz, zRy}	{xRz, zRx}	Yes

There are 5 non-trivial relationships from the above table:

1) If {xRy, yRx} and yRz, then xRz.

2) If {xRy, yRx} and zRy, then zRx.

3) If xRy and {yRz, zRy}, then xRz.

4) If yRx and {yRz, zRy}, then zRx.

5) If {xRy, yRx} and {yRz, zRy}, then {xRz, zRx}.

These 5 relationships along with the relationship from Axiom 2 define transitivity when ties are taken into consideration. Since
xIy ≡ {xRy, yRx} by Arrow, we can write the following transitive relationships.

1) xRyRz

2) xIyRz

3) zRxIy

4) xRyIz

5) yRxIz

6) xIyIz

Any permutation of x,y and z is allowed in the above relationships. These taken together with the trivial relationships are the only relationships allowed by a generalized version of Axiom 2. Instead of Arrow's definition of indifference given above we can define a tie operator, T, such that xTy ≡ {xRy, yRx}. This would be a more general relationship way of defining {xRy, yRx} than as an indifference.

A generalized version of Axiom 2, then, limits the relationships among three alternatives to the ones given above which can be expressed as alternative operator alternative operator alternative operator. Any relationship among three variables that cannot be expressed in this way is intransitive.

For three alternatives, Axiom 1 needs to be generalized also. There are six possible outcomes not involving ties: xRyRz, xRzRy, yRxRz, yRzRx, zRxRy, zRyRx. These correspond to xRy and yRx in the binary case. In order to be complete, we need to include all combinations of these outcomes as possible outcomes similar to the binary case.

Axiom 1 (generalized): For all x, y, z, either xRyRz or xRzRy or yRxRz or yRzRx or zRxRy or zRyRx or combinations of the type {xRyRz, xRzRy} or combinations of the type {xRyRz, xRzRy, yRxRz} or combinations of the type {xRyRz, xRzRy, yRxRz, yRzRx} or combinations of the type {xRyRz, xRzRy, yRxRz, yRzRx, zRxRy} or {xRyRz, xRzRy, yRxRz, yRzRx, zRxRy, zRyRx} where is the binomial coefficient

The Social Welfare Function (SWF)

Arrow defines a SWF as follows:

Definition 4: By a social welfare function will be meant a process or rule which, for each set of individual orderings $R_1, ..., R_n$ for alternative social states (one ordering for each individual), states a corresponding social ordering of alternative social states, R.

Therefore, the scenario is that each individual voter/consumer specifies either xR_iy or yR_ix or possibly, for the sake of completeness, $\{xR_iy, yR_ix\}$. There is absolutely no way to determine whether or not a voter *prefers* x to y or y to x because his specification is that he "prefers or is indifferent" between x and y. We cannot ever conclude that not yR_ix is xP_iy because we never know particular preference and indifference information—only "preference or indifference" information. We know from Axiom 1 that either xR_iy or yR_ix or both which we write $\{xR_iy, yR_ix\}$. Either we have xR_iy or yR_ix or a tie. Arrow goes on to define a tie as an indifference. In any case, if yR_ix is not true, then xR_iy or $\{xR_iy, yR_ix\}$ must be true. We can never get preference information out of the individual orderings $R_1, ..., R_n$ since it's not called for in Definition 4.

Even with a formal logical consideration, preference information is not available. Let's define xRy formally as xPy OR xIy where OR is logical or. Then if each individual specifies according to Axiom 1 either 1) xR_iy, 2) yR_ix or 3) $\{xR_iy, yR_ix\} \equiv xI_iy$, one of the above three must be true and the other two false. Therefore, if NOT yR_ix is true where NOT is the logical not, then either xR_iy or xI_iy must be true. If individual i specifies xR_iy, one cannot conclude xP_iy; one can only conclude that i prefers x to y or is indifferent between x and y. We can only conclude that xP_iy if we know that xR_iy AND {NOT xI_iy} where AND is the logical and. But by Definition 4 and Axiom 1, we do not elicit that information from individual i. We only elicit 1), 2) or 3) above and not {(2) AND NOT (3)}.

Even without ties, NOT $yR_ix \Rightarrow xP_iy$ is not true. If we assume that each individual can only specify xR_iy or yR_ix and cannot specify $\{xR_iy, yR_ix\} \equiv xI_iy$, we still cannot extract preference information. If y is not preferred or indifferent to x, y can still be indifferent to x. Therefore, NOT yR_ix does not imply xP_iy. The truth table is the following:

yR_ix	NOT yR_ix	yP_ix	xP_iy	xI_iy
0	1	0	0	1
0	1	0	1	0

Considering the SCF and Definition 4, if the SWF were xRyRz, the SCF would be x. If the SWF were {xRy, yRx}Rz or in Arrow's notation xIyRz, the SCF would be [x,y]. The SCF truncates the SWF and yields the top alternative or set of alternatives in case the top alternatives are tied. With three alternatives it is possible in accordance with generalized Axiom 1 to have solutions of the form {xRzRy, zRyRx} or {zRyRx, yRxRz, yRzRx}. This generalizes directly from the binary case, {xRy, yRx}. The most general tie with three alternatives would be {xRyRz, xRzRy, yRxRz, yRzRx, zRxRy, zRyRx}.

With generalized Axiom 1, a tie refers to *orderings* and not to *alternatives*. The choice function C(S) would specify a tie between the alternatives x and y if xIyRz were the social ordering, for example. We are considering here ties among the orderings themselves and not just among the "top slot" of those orderings.

Let us assume there are six voter/consumers and that each specifies a different ordering. There is then one specification for each possible individual ordering. The common sense, heuristic solution is a tie among all the possible orderings. Similarly, there are 24 possible non-tie orderings for 4 alternatives, and, for the case of 24 voter/consumers, each specifying a different ordering, common sense would dictate a tie among all the possible social orderings. A similar case can be made for m=5, 6, ... These are the broadest conceivable tie sets, and will be called maximal tie sets. Tie sets involving less than the total number of orderings are also possible and demanded by the completeness requirement.

Independence of Irrelevant Alternatives

Arrow postulates five conditions with which, in addition to Axioms 1 and 2, a SWF should be compliant. The only one of these that involves the SCF is Condition 3, the Independence of Irrelevant Alternatives.

Condition 3: "Let R_1, ..., R_n and R'_1, ..., R'_n be two sets of individual orderings and let C(S) and C'(S) be the corresponding social choice functions. If, for all individuals i and x and y in a given environment S, xR_iy if and only if xR'_iy, then C(S) and C'(S) are the same."

Arrow postulates that only the top slot of the orderings in environment S be the same since C(S) only selects the top slot. It might be asked why it shouldn't be required that the top two slots of the orderings be the same or the top three slots etc. Therefore, for a general SWF solution such as

$$x_1Rx_2R \ldots x_jRx_{j+1}R \ldots x_{m-1}Rx_m$$

we can generalize the SCF as follows:

$C_1 = x_1$

$C_2 = x_1Rx_2$

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$C_j = x_1Rx_2R \ldots Rx_j$

C_j is the ordering of the top j alternatives. In particular, if there are s alternatives in the environment S, then C_s would be the ordering of all those alternatives. C_1 = C(S), Arrow's social choice function.

As an example, let us assume that R corresponding to $R_1, \ldots, R_n$ is uRvRwRxRyRz and R' corresponding to $R'_1, \ldots, R'_n$ is uRvRyRxRwRz and that S = {v, w, x, y}. Also all individuals vote exactly the same for v, w, x and y in the two sets of individual orderings. Then Arrow's Condition 3 is met since C(S) = C'(S) = v. However, $C_2(S)$ = vRw ≠ $C'_2(S)$ = vRy. Similarly, $C_3(S)$ ≠ $C'_3(S)$ and $C_4(S)$ ≠ $C'_4(S)$. Arrow's Condition 3 is the weakest possible requirement!

Consider another example. Let us assume that R corresponding to $R_1, \ldots, R_n$ is uRvRwRxRyRz and R' corresponding to $R'_1, \ldots, R'_n$ is zRvRwRxRyRu and that S = {v, w, x, y}. Also all individuals vote exactly the same for v, w, x and y in the two sets of individual orderings. Then C(S) = C'(S) = v. $C_2(S)$ = $C'_2(S)$ = vRw. $C_3(S)$ = $C'_3(S)$ = vRwRx and $C_4(S)$ = $C'_4(S)$ = vRwRxRy.

When ties are considered, Arrow's social choice function is inadequate. Consider the following example. R = {xRyRz, yRzRx, zRxRy}. R' = {zRxRy, yRzRx, xRzRy}. Let S = {x,y} and assume that all individuals vote exactly the same for x and y in the two cases. In S we have R = {xRy, yRx, xRy} and R' = {xRy, yRx, xRy}, and C(S) = C'(S) since R and R' are identical. However, there is no way of determining what C(S) is unless an additional rule is laid down for combining terms in the expression {xRy, yRx, xRy}. Let us use the following rule: For m = 3 and S = {x,y}, let $C_2(S)$ = the maximum over R of xRy and yRx. In the tie expression above, {xRy, yRx, xRy}, there are 2 xRys and 1 yRx. Therefore, $C_2(S)$ = xRy.

The generalized SCF then consists of orderings and a rule for selecting these orderings when the SWF is singular or a tie.

The Ternary Case — n odd

We now proceed to demonstrate solutions which are social orderings for a specific SWF for the case $m = 3$ which satisfy a strengthened version of Arrow's conditions. Let us assume alternatives x, y and z and n (odd) voter/consumers. As a consequence of Arrow's Condition 3, the independence of irrelevant alternatives, we know that "knowing the social choices made in pairwise comparisons determines the entire social ordering." Accordingly, we consider the social choices of the alternatives two by two. Our SWF is as follows. If $N(x,y) > N(y,x)$, then xRy. If $N(y,x) > N(x,y)$, then yRx. At the ternary level we have 8 cases:

Case 1: xRy, xRz, yRz

Case 2: xRy, xRz, zRy

Case 3: xRy, zRx, yRz

Case 4: xRy, zRx, zRy

Case 5: yRx, xRz, yRz

Case 6: yRx, xRz, zRy

Case 7: yRx, zRx, yRz

Case 8: yRx, zRx, zRy

According to the Condorcet (1785) method for determining the outcome of an election, we consider each of the alternatives in pairs, determine the winner for each pair and then determine the final social ordering by combining these results. We use the Condorcet method in our SWF for the above cases in which it actually produces a result. Therefore, we have the following:

Case	Social Ordering
1	xRyRz
2	xRzRy
4	zRxRy
5	yRxRz
7	yRzRx
8	zRyRx

This leaves only cases 3 and 6. Consider the solution {xRyRz, yRzRx, zRxRy} for Case 3. We call a reduced ordering or reduced solution an ordering with one or more alternatives removed. If we consider {xRyRz, yRzRx, zRxRy} and remove z, we get {xRy, yRx, xRy}. Combining terms we have {2xRy, yRx}. If we choose the most numerous of xRy and yRx as the solution, we get xRy by 2 to 1 which is in accordance with the binary result.

Likewise, if we reduce {xRyRz, yRzRx, zRxRy} by y, we get {xRz, zRx, zRx} or {xRz, 2zRx}. 2zRx > xRz and we take zRx as the reduced solution which agrees with the known binary solution. Similarly, if we remove x from the social solution, we have {yRz, yRz, zRy} which yields yRz. Accordingly, our SWF algorithm is as follows:

1) Choose the Condorcet solution if it exists.

2) If the Condorcet solution doesn't exist, construct a solution such that, when the solution is reduced by any single alternative, the most numerous of the remaining binary relationships is the same as the known binary solution.

Notice that our algorithm will always produce consistent results if the ternary solution is generated from the binary solution in such a way that there is a 2 to 1 ratio between the correct binary solution and the incorrect binary solution and then we take the larger of the two as our reduced solution. We construct our solutions in this manner in order to be compliant with Arrow's Condition 3. Satisfying the other Conditions is then trivial and is easily shown. Whether or not such a solution always exists for m > 3 will be answered affirmatively elsewhere (Lawrence, 1998). Here all we need to show is the existence of a solution for Case 6. Consider the solution {yRxRz, xRzRy, zRyRx}. Reduction by z yields yRx; by y, xRz; by x, zRy. These all agree with the known binary cases and are consistent with the above definition.

Therefore, we have demonstrated a consistent algorithm for the SWF which yields the same social orderings when reduced from the ternary case to the binary case as those produced at the binary level directly from the domain. There is complete consistency of social orderings and not just of alternatives produced by the choice function. The choice function only produces the top position in an ordering. We demand consistency over all orderings which can be produced by reducing a social ordering and this strengthens Arrow's Condition 3.

Arrow (1951, p. 26) states that “...suppose that an election system has been devised whereby each individual lists all the candidates in order of his preference and then, by a preassigned procedure, the winning candidate is derived from these lists. ...Suppose an election is held, with a certain number of candidates in the field, each individual filing his list of preferences, and then one of the candidates dies. Surely the social choice should be made by taking each of the individual's preference lists, blotting out completely the dead candidate's name, and considering only the orderings of the remaining names in going through the procedure of determining a winner.” This is precisely what we have done in choosing our SWF. Notice that it is completely consistent with the solutions for those cases determined by the Condorcet method and yields consistent results when the “dead” candidate is “blotted out” of the social ordering as well as when the “dead” candidate is blotted out of the individual orderings.

Let us examine the result produced by the SCF function for the above cases. Let S ={x,y}

Case	Social Ordering	C_1	C_2
1	xRyRz	x	xRy
2	xRzRy	x	xRy
3	{xRyRz, yRzRx, zRxRy}	x	xRy
4	zRxRy	z	xRy
5	yRxRz	y	yRx
6	{yRxRz, xRzRy, zRyRx}	y	yRx
7	yRzRx	y	yRx
8	zRyRx	y	yRx

Now let’s consider Arrow’s Condition 3 and let R_i be such as to produce xRy, yRz, xRz (Case 1) and R'_i be such as to produce xRy, yRz zRx (Case 3). S = {x,y}. Then $C_1(S)$ should equal $C'_1(S)$. It does and also $C_2(S) = C'_2(S)$ which is more than Arrow requires. Similarly, all the above cases satisfy not only Arrow’s Condition 3 which requires C_1 but a stronger condition requiring C_2.

Examining the above relationships we notice that, regardless of the relationship between x and z and y and z, as long as xRy, C_1 = x and C_2 = xRy.

The Ternary Case — n even

When n is even we have a total of 27 cases. We have already considered the first 8 cases above. For convenience we define {xRy, yRx} as xTy instead of xIy since it is more general. In addition there is one more tie possibility, a three way tie: N(x,y) = N(y,x) = N(y,z) = N(z,y) = N(x,z) = N(z,x). We write this as {xRy, yRx, yRz, zRy, xRz, zRx} and define this as xTyTz. Solutions for the remaining cases are shown below. We also note the following transitivity requirements:

1) If xRy and yTz, then xRz;
2) If xTy and yRz, then xRz;
3) If xTy and yTz, then xTz.

Case	Binary Solutions	Ternary Solution
9:	xRy, xRz, yTz	xRyTz
10:	xRy, zRx, yTz	{zRxRy, xRyTz, yTzRx}
11:	yRx, xRz, yTz	{yRxRz, xRyTz, yTzRx}
12:	yRx, zRx, yTz	yTzRx
13:	xRy, xTz, yRz	{xRyRz, yRxTz, xTzRy}
14:	xRy, xTz, zRy	xTzRy
15:	yRx, xTz, yRz	yRxTz
16:	yRx, xTz, zRy	{zRyRx, yRxTz, xTzRy}
17:	xTy, xRz, yRz	xTyRz
18:	xTy, xRz, zRy	{xRzRy, xTyRz, zRxTy}
19:	xTy, zRx, yRz	{yRzRx, xTyRz, zRxTy}
20:	xTy, zRx, zRy	zRxTy
21:	xRy, xTz, yTz	{xRyTz, xTzRy, xTyTz}
22:	yRx, xTz, yTz	{yRxTz, yTzRx, xTyTz}
23:	xTy, xRz, yTz	{xRyTz, xTyRz, xTyTz}
24:	xTy, zRx, yTz	{zRxTy, yTzRx, xTyTz}
25:	xTy, xTz, yRz	{yRxTz, xTyRz, xTyTz}
26:	xTy, xTz, zRy	{zRxTy, xTzRy, xTyTz}
27:	xTy, xTz, yTz	xTyTz

Now, if xRy and S = {x,y}, we require $C_2 = C'_2$ regardless of the relationship between x and z and y and z. An examination of the above cases shows this to be true.

m = 4 — n odd

For m = 4 and no ties considered (n odd), we have the following solutions. (Note that abcd ≡ aRbRcRd.) The generalized SCF amounts to starting with the result produced by the SWF and reducing the solution down to the set S where $S \subset T$ according to the above algorithm. This is completely in accordance with starting with the set S and generating a result according to the above SWF. In other words we can reduce down to S or expand up to S with the same results using the algorithm presented here. The results produced are identical and $C_j(S) = C'_j(S)$ in every case for $1 \le j \le 3$.

Case 1: aRb, aRc, aRd, bRc, bRd , cRd Solution: abcd

Case 2: aRb, aRc, aRd, bRc, bRd, dRc Solution: abdc

Case 3: aRb, aRc, aRd, bRc, dRb, cRd Solution: abcd, acdb, adbc

Case 4: aRb, aRc, aRd, bRc, dRb, dRc Solution: adbc

Case 5: aRb, aRc, aRd, cRb, bRd , cRd Solution: acbd

Case 6: aRb, aRc, aRd, cRb, bRd , cRd Solution: abdc, acbd, adcb

Case 7: aRb, aRc, aRd, cRb, dRb , cRd Solution: acdb

Case 8: aRb, aRc, aRd, cRb, dRb , dRc Solution: adcb

Case 9: aRb, aRc, dRa, bRc, bRd , cRd Solution: abcd, bcda, dabc

Case 10: aRb, aRc, dRa, bRc, bRd , dRc Solution: abdc, bdac, dabc

Case 11: aRb, aRc, dRa, bRc, dRb, cRd Solution: abcd, dabc, cdab

Case 12: aRb, aRc, dRa, bRc, dRb, dRc Solution: dabc

Case 13: aRb, aRc, dRa, cRb, bRd , cRd Solution: acbd, cbda, dacb

Case 14: aRb, aRc, dRa, cRb, bRd , dRc Solution: acbd, bdac, dacb

Case 15: aRb, aRc, dRa, cRb, dRb , cRd Solution: acdb, cdab, dacb

Case 16: aRb, aRc, dRa, cRb, dRb , dRc Solution: dacb

Case 17: aRb, cRa, aRd, bRc, bRd , cRd Solution: abcd, bcad, cabd

Case 18: aRb, cRa, aRd, bRc, bRd , dRc Solution: abdc, bdca, cabd

Case 19: aRb, cRa, aRd, bRc, dRb , cRd Solution: adbc, bcad, cadb

Case 20: aRb, cRa, aRd, bRc, dRb , dRc Solution: adbc, dbca, cadb

Case 21: aRb, cRa, aRd, cRb, bRd , cRd Solution: cabd

Case 22: aRb, cRa, aRd, cRb, bRd , dRc Solution: cabd, abdc, dcab

Case 23: aRb, cRa, aRd, cRb, dRb , cRd Solution: cadb

Case 24: aRb, cRa, aRd, cRb, dRb , dRc Solution: cadb, dcab, adcb

Case 25: aRb, cRa, dRa, bRc, bRd , cRd Solution: abcd, bcda, cdab

Case 26: aRb, cRa, dRa, bRc, bRd , dRc Solution: abdc, bdca, dcab

Case 27: aRb, cRa, dRa, bRc, dRb , cRd Solution: cdab, bcda, dabc

Case 28: aRb, cRa, dRa, bRc, dRb , dRc Solution: dcab, dbca, dabc

Case 29: aRb, cRa, dRa, cRb, bRd , cRd Solution: cabd, cbda, cdab

Case 30: aRb, cRa, dRa, cRb, bRd , dRc Solution: cabd, bdca, dcab

Case 31: aRb, cRa, dRa, cRb, dRb , cRd Solution: cdab

Case 32: aRb, cRa, dRa, cRb, dRb , dRc Solution: dcab

Case 33: bRa, aRc, aRd, bRc, bRd , cRd Solution: bacd

Case 34: bRa, aRc, aRd, bRc, bRd , dRc Solution: badc

Case 35: bRa, aRc, aRd, bRc, dRb , cRd Solution: acdb, bacd, dbac

Case 36: bRa, aRc, aRd, bRc, dRb , dRc Solution: adbc, badc, dbac

Case 37: bRa, aRc, aRd, cRb, bRd , cRd Solution: acbd, bacd, cbad

Case 38: bRa, aRc, aRd, cRb, bRd , dRc Solution: adcb, badc, cbad

Case 39: bRa, aRc, aRd, cRb, dRb , cRd Solution: acdb, bacd, cdba

Case 40: bRa, aRc, aRd, cRb, dRb , dRc Solution: adcb, badc, dcba

Case 41: bRa, aRc, dRa, bRc, bRd , cRd Solution: bacd, bcda, bdac

Case 42: bRa, aRc, dRa, bRc, bRd , dRc Solution: bdac

Case 43: bRa, aRc, dRa, bRc, dRb , cRd Solution: bacd, cdba, dbac

Case 44: bRa, aRc, dRa, bRc, dRb , dRc Solution: dbac

Case 45: bRa, aRc, dRa, cRb, bRd , cRd Solution: acbd, bdac, cbda

Case 46: bRa, aRc, dRa, cRb, bRd , dRc Solution: dacb, bdac, cbda

Case 47: bRa, aRc, dRa, cRb, dRb , cRd Solution: acdb, dbac, cdba

Case 48: bRa, aRc, dRa, cRb, dRb , dRc Solution: dacb, dbac, dcba

Case 49: bRa, cRa, aRd, bRc, bRd , cRd Solution: bcad

Case 50: bRa, cRa, aRd, bRc, bRd , dRc Solution: badc, bcad, bdca

Case 51: bRa, cRa, aRd, bRc, dRb , cRd Solution: cadb, bcad, dbca

Case 52: bRa, cRa, aRd, bRc, dRb , dRc Solution: adbc, bcad, dbca

Case 53: bRa, cRa, aRd, cRb, bRd , cRd Solution: cbad

Case 54: bRa, cRa, aRd, cRb, bRd , dRc Solution: badc, cbad, dcba

Case 55: bRa, cRa, aRd, cRb, dRb , cRd Solution: cadb, cbad, cdba

Case 56: bRa, cRa, aRd, cRb, dRb , dRc Solution: adcb, cbad, dcba

Case 57: bRa, cRa, dRa, bRc, bRd , cRd Solution: bcda

Case 58: bRa, cRa, dRa, bRc, bRd , dRc Solution: bdca

Case 59: bRa, cRa, dRa, bRc, dRb , cRd Solution: bcda, cdba, dbca

Case 60: bRa, cRa, dRa, bRc, dRb , dRc Solution: dbca

Case 61: bRa, cRa, dRa, cRb, bRd , cRd Solution: cbda

Case 62: bRa, cRa, dRa, cRb, bRd , dRc Solution: bdca, dcba, cbda

Case 63: bRa, cRa, dRa, cRb, dRb , cRd Solution: cdba

Case 64: bRa, cRa, dRa, cRb, dRb , dRc Solution: dcba

Let S_1 = {a}, S_2 = {a, b}, S_3 = {a, b, c}. Then $C_1(S_1) = C'_1(S_1)$; $C_2(S_2) = C'_2(S_2)$; $C_3(S_3) = C'_3(S_3)$ for every possible case above by inspection. For example, consider Cases 59 and 60 as R and R', respectively. $C_1(S_1) = C'_1(S_1)$ = a; $C_2(S_2) = C'_2(S_2)$ = bRa; $C_3(S_3) = C'_3(S_3)$ = bRcRa.

Conclusion

A generalized SCF was developed in which orderings can be extracted from a SWF for a set $S \subset T$ where the set T contains all m alternatives under consideration. The relationship between the SCF and the SWF was examined, and it was shown that solutions produced by a particular SWF for S are the same as the solutions produced by the highest order, generalized SCF for S. In other words, using the SWF and generating solutions for $S \subset T$, yields the same results as taking the solutions produced by the SWF for the set T and applying the generalized SCF.

Arrow claims to provide for tie solutions using the SCF. However, the SCF can only produce ties among alternatives. Since the purpose of the SWF is to produce orderings, it would seem natural to explore the possibility of ties among orderings. In his Axiom 1 Arrow allows for ties between orderings for two alternatives. We have generalized this to allow for ties among orderings of three or more alternatives.

We have also generalized the axioms of transitivity and completeness to take into account tie solutions. If a solution produced by a SWF is a tie, then further measures can be taken to winnow the solution set to a singular solution at least in some cases. This is the subject of another paper and was not considered here.

We have pointed out several errors in Arrow's logic—in particular involving his assertion that NOT yRx $\Rightarrow$ xPy which is untrue. The logical errors throw into question Arrow's entire analysis which, therefore, should not be considered cast in concrete.

We have shown that, when ties are properly considered, a SWF exists for m = 3 which complies with a strengthened version of Arrow's Condition 3, the Independence of Irrelevant Alternatives. Elsewhere (Lawrence 1998), it has been shown that this result complies with Arrow's other conditions and generalizes for any value of m and n. Therefore, social choice or, more precisely, a SWF does exist when ties among orderings are considered correctly.

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