Neutrality and the Possibility of Social Choice

by

John Clifton Lawrence
General Algorithm
POB 230351
Encinitas, CA 92023
Phone/fax: 760-633-3778
Web site: http://www.genalg.com
E-mail: jlawrence@genalg.com

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PO Box 126155
San Diego, CA 92112
Website: http://www.socialchoiceandbeyond.com
Email: j.c.lawrence@cox.net
Blog: http://willblogforfood.typepad.com

Abstract

In “Social Choice and Individual Values,” Kenneth Arrow (1951) postulates five criteria that a Social Welfare Function (SWF) must comply with, in his opinion, in order to be rational and ethical. One of these is Citizens’ Sovereignty which states that the social decision must be a function of the individuals’ preference information and nothing else. A strengthened version of this is the Principle of Neutrality which states that the SWF must yield the same social solution with regard to two alternatives x and y with x and y interchanged if x and y are interchanged in all the individual preference data. For the case of two alternatives, Arrow proves that social choice is possible only by virtue of the fact that he violates the Principle of Neutrality (while complying with Citizens’ Sovereignty) and treats x and y differently when half the voters prefer y to x and half prefer x to y. The way Arrow treats this tie case leads to the conclusion that social choice is possible in the case of m (the number of alternatives) = 2 whereas a treatment honoring the Principle of Neutrality and without the consideration of ties would lead to the conclusion that social choice is impossible for m = 2 also. If ties are considered legitimate for m = 2, then social choice which honors the Principle of Neutrality is possible for m = 2. This then opens the door to the possibility of social choice for m = 3, 4... .

Introduction

We consider the binary case of two alternatives, x and y, and n voters. We assume that each individual has a preference ordering over the alternatives as indicated by xP_iy or yP_ix where the subscript i represents the i^{th} individual. In general xQ_iy, where Q_i is chosen from the set {P_i, not P_i} The corresponding social orderings are represented by P and Q (without the subscripts) so that either xPy or yPx and xQy with $Q = f(Q_1, Q_2, \ldots Q_n)$. The Social Welfare Function (SWF) assigns elements of the range to elements of the domain $(Q_1, Q_2, \ldots Q_n)$. Later we will introduce the concepts of "indifference," I, and "preference or indifference," R, but for now let's examine the case of two alternatives, x and y, and the relationship, P.

If n is an even number and n/2 voters specify xP_iy while the other n/2 voters specify yP_ix, then we clearly have a tie which we indicate {xPy, yPx}. Note that P does not have to be reflexive for this voting rule to be perfectly rational, but it does have to be complete [i.e.] either xPy, yPx or {xPy, yPx}. These then comprise the set of range elements that can be considered social orderings. Heuristically and intuitively, we *must* provide for the possibility of a tie as a valid range option. We know from experience that such an outcome is possible. Yet Arrow does not consider ties in his discussion of the binary case which we will discuss further below.

If we consider both P and I, then the individual voter/consumers specify xP_iy, yP_ix or $xI_iy = yI_ix$. The corresponding social orderings are xPy, yPx, xIy, {xPy, yPx}, {xPy, xIy}, {yPx, xIy}, {xPy, yPx, xIy}. xIy might heuristically be appropriate if the majority of voters are indifferent between x and y but not appropriate if half the voters prefer x to y and half, y to x. For the sake of completeness, both solutions are available. Note that the domain in the P and I world *includes* the domain of the P world. Since I is reflexive and P is not reflexive, some of the individual orderings are reflexive, namely xI_iy and only one social ordering is reflexive: xIy.

We examine a particular SWF known as majority rule which can be defined as follows:

MAJORITY RULE (P): Let N(x,y) be the number of individuals who prefer x to y and N(y,x) be the number of individuals who prefer y to x. Then, if N(x,y) > N(y,x), xPy; if N(y,x) > N(x,y), yPx; and, if N(x,y) = N(y,x), there is a tie indicated by {xPy, yPx}.

The above definition of majority rule satisfies Arrow's 5 Criteria and rationality axioms. The rationality axioms require P to be complete (Axiom 1) and transitive (Axiom 2). We rephrase Axiom 1 as follows:

Axiom 1: For all x and y ≠ x, either xPy, yPx or both.

Notice that P is not reflexive i.e. xPx is not true. However, to require reflexivity would not be rational in this context nor is reflexivity necessary to prove that majority rule for P meets Arrow's criteria since various writers [notably Sen (1970)] have proven Arrow's Impossibility Theorem without using R (which is reflexive) and only using P and I (which taken together aren't). Axiom 2 which requires transitivity only applies when there are 3 or more alternatives and P is transitive since if xPy and yPz, then xPz.

ARROW'S CONDITION 1: **Unrestricted domain**

The domain is entirely unrestricted since a solution is provided for all possible combinations of individual preferences. Note, however, that there are only 3 possible cases regardless of how the individuals state their preferences: N(x,y) > N(y,x); N(y,x) > N(x,y); and N(x,y) = N(y,x).

ARROW'S CONDITION 2: **Positive Association of Social and Individual Values**

This condition requires that, if every individual raises some candidate in his preference list, that candidate must not be lowered in the social choice. The majority rule considered here satisfies an even stronger criterion which is, if any individual raises a candidate in his preference list, the preference lists of all other voters remaining the same, then that candidate will either be raised or stay the same in the social choice.

Proof: There are three cases. Case 1: N(x,y) > N(y,x). If one voter changes his vote from yP_ix to xP_iy, then the social ordering is still xPy and the condition is satisfied. If one voter changes his vote from xPiy to yPix, either N(x,y) > N(y,x) still or N(x,y) = N(y,x) (if n even) or N(y,x) > N(x,y) (if n odd). If N(x,y) = N(y,x), the social ordering changes from xPy to a tie between xPy and yPx and y has been elevated in the social ordering from a loss to a tie. If N(y,x) > N(x,y), the social ordering changes from xPy to yPx and y has been elevated in the social ordering from a loss to a win. Case 2: N(y,x) > N(x,y). Same as Case 1. Case 3: N(x,y) = N(y,x). If one voter changes his vote from yPix to xPiy, then the social ordering changes from a tie between xPy and yPx to xPy and x has been elevated in the social ordering. Exactly the converse is true if one voter changes his vote from xPiy to yPix.

ARROW'S CONDITION 3: **Independence of Irrelevant Alternatives**

This condition states that, given two different sets of individual orderings, the social choice from any two identical subsets of alternatives must be the same if the individual orderings for those two subsets are identical regardless of the individual orderings among the other alternatives.

For the binary case this is trivial since the only subset of two alternatives is one alternative. Therefore, the choice from a subset of one alternative will always be that alternative regardless of what the two different sets of individual orderings are.

ARROW'S CONDITION 4: **Citizens' Sovereignty**

Citizens' Sovereignty obtains if, for no pair of distinct alternatives x and y, xPy for every set of individual orderings P_1, P_2, ... P_n.

Citizens' Sovereignty would not obtain if, regardless of the data, the social ordering is always the same. Neutrality is a stronger condition that requires that all alternatives be considered similarly by the SWF. The SWF is neutral for two alternatives if, for any set of individual data, P_1, P_2, ... P_n, and xPy, the social ordering will be yPx if x and y are interchanged in each individual's list. Furthermore, taking ties into account, one would expect that, for any set of individual data and {xPy, yPx}, if x and y are interchanged in each individual's list, the social ordering remains {xPy, yPx}.

Both Citizens' Sovereignty and Neutrality are clearly satisfied by Majority Rule (P).

ARROW'S CONDITION 4: **Nondictatorship**

A SWF is dictatorial if there exists an individual i such that, for all x and y, xP_iy implies xPy regardless of the orderings P_1, P_2, ... P_n of all individuals other than i. A weaker version of this is dictatorship over one issue: for *some* x and y, xP_iy implies xPy regardless of the orderings P_1, P_2, ... P_n of all individuals other than i. A stronger version of Nondictatorship is Anonymity which requires that all voters be treated exactly the same in the voting process regardless of their identity.

Majority Rule (P) is clearly non-dictatorial.

Preferences and Indifferences

Now each individual can prefer x to y, xP_iy; prefer y to x, yP_ix or be indifferent between x and y, xI_iy. Similarly, (assuming no ties for now) society can prefer x to y, xPy; prefer y to x, yPx; or be indifferent between x and y, xIy. We can characterize the domain by N(x,y), the number who prefer x to y; N(y,x), the number who prefer y to x; and M(x,y), the number who are indifferent between x and y.

There are now 13 cases which characterize the domain:

1) $N(x,y) > N(y,x) > M(x,y)$
2) $N(x,y) > M(x,y) > N(y,x)$

3) N(y,x) > N(x,y) > M(x,y)

4) N(y,x) > M(x,y) > N(x,y)

5) M(x,y) > N(x,y) > N(y,x)

6) M(x,y) > N(y,x) > N(x,y)

7) N(x,y) > N(y,x) = M(x,y)

8) N(y,x) > N(x,y) = M(x,y)

9) M(x,y) > N(x,y) = N(y,x)

10) N(x,y) = N(y,x) > M(x,y)

11) N(x,y) = M(x,y) > N(y,x)

12) N(y,x) = M(x,y) > N(x,y)

13) N(x,y) = N(y,x) = M(x,y)

Correspondingly, including ties, there are 7 possibilities for the range: xPy, yPx, xIy, {xPy, yPx}, {xPy, xIy}, {yPx, xIy}, {xPy, yPx, xIy}. A SWF would assign one of the range possibilities to each of the domain cases. Among possible assignments, there are at least three forms of majority rule. The first [**Majority Rule (PI1)**] is given by the following assignment in which individual indifferences are simply ignored:

Domain Case		**SWF Assignment**
1)	N(x,y) > N(y,x) > M(x,y)	xPy
2)	N(x,y) > M(x,y) > N(y,x)	xPy
3)	N(y,x) > N(x,y) > M(x,y)	yPx
4)	N(y,x) > M(x,y) > N(x,y)	yPx
5)	M(x,y) > N(x,y) > N(y,x)	xPy
6)	M(x,y) > N(y,x) > N(x,y)	yPx
7)	N(x,y) > N(y,x) = M(x,y)	xPy
8)	N(y,x) > N(x,y) = M(x,y)	yPx
9)	M(x,y) > N(x,y) = N(y,x)	{xPy,yPx}
10)	N(x,y) = N(y,x) > M(x,y)	{xPy,yPx}
11)	N(x,y) = M(x,y) > N(y,x)	xPy

12)	N(y,x) = M(x,y) > N(x,y)	yPx
13)	N(x,y) = N(y,x) = M(x,y)	{xPy,yPx}

The second [**Majority Rule (PI2)**] is given by the following assignment in which individual indifferences are ignored and ties are equated to indifferences:

Domain Case		**SWF Assignment**
1)	N(x,y) > N(y,x) > M(x,y)	xPy
2)	N(x,y) > M(x,y) > N(y,x)	xPy
3)	N(y,x) > N(x,y) > M(x,y)	yPx
4)	N(y,x) > M(x,y) > N(x,y)	yPx
5)	M(x,y) > N(x,y) > N(y,x)	xPy
6)	M(x,y) > N(y,x) > N(x,y)	yPx
7)	N(x,y) > N(y,x) = M(x,y)	xPy
8)	N(y,x) > N(x,y) = M(x,y)	yPx
9)	M(x,y) > N(x,y) = N(y,x)	xIy
10)	N(x,y) = N(y,x) > M(x,y)	xIy
11)	N(x,y) = M(x,y) > N(y,x)	xPy
12)	N(y,x) = M(x,y) > N(x,y)	yPx
13)	N(x,y) = N(y,x) = M(x,y)	xIy

The third [**Majority Rule (PI3)**] is given by the following assignment in which the most numerous group is declared the winner and ties are included:

	Domain Case	**SWF Assignment**
1)	N(x,y) > N(y,x) > M(x,y)	xPy
2)	N(x,y) > M(x,y) > N(y,x)	xPy
3)	N(y,x) > N(x,y) > M(x,y)	yPx
4)	N(y,x) > M(x,y) > N(x,y)	yPx
5)	M(x,y) > N(x,y) > N(y,x)	xIy
6)	M(x,y) > N(y,x) > N(x,y)	xIy
7)	N(x,y) > N(y,x) = M(x,y)	xPy
8)	N(y,x) > N(x,y) = M(x,y)	yPx
9)	M(x,y) > N(x,y) = N(y,x)	xIy
10)	N(x,y) = N(y,x) > M(x,y)	{xPy,yPx}
11)	N(x,y) = M(x,y) > N(y,x)	{xPy,xIy}
12)	N(y,x) = M(x,y) > N(x,y)	{yPx,xIy}
13)	N(x,y) = N(y,x) = M(x,y)	{xPy,yPx,xIy}

All of these varieties of majority rule clearly satisfy Arrow's 5 criteria.

Preferences or Indifferences—the R Operator

The R operator is defined as "prefered or indifferent to". xR_iy means that the i^{th} individual prefers x to y or is indifferent between x and y. Similarly, xRy means that society prefers or is indifferent between x and y. If an individual specifies xR_iy, we don't know if he actually prefers x to y or is indifferent between the two alternatives. We only know that one or the other is true. The same can be said for society. Let **N(x,y)** be the number of individuals who "prefer or are indifferent between" x and y and **N(y,x)** be the number of individuals who "prefer or are indifferent between" y and x. Then there are three cases for the domain: **N(x,y)** > **N(y,x)**, **N(y,x)** > **N(x,y)** and { **N(x,y)**, **N(y,x)**}. This is formally the same situation as was analyzed for the P operator. Correspondingly, there are three range assignments: xRy, yRx and {xRy, yRx}.

Therefore, the majority rule for the R operator, similar to that stated above for the P operator, can be stated as follows:

MAJORITY RULE (R): Let **N(x,y)** be the number of individuals who prefer or are indifferent between x and y and **N(y,x)** be the number of individuals who prefer or are indifferent between y and x. Then, if **N(x,y)** > **N(y,x)**, xRy; if **N(y,x)** > **N(x,y)**, yRx; and, if **N(x,y)** = **N(y,x)**, there is a tie indicated by {xRy, yRx}.

Arrow's Treatment of Ties

Arrow's Axiom I states: "For all x and y, either xRy or yRx." He goes on to state (p. 13): "Note also that the word 'or' in the statement of Axiom I does not exclude the possibility of both xRy and yRx. That word merely asserts that at least one of the two events must occur; both may." If both events occur, Arrow implies, this event would be considered a tie: a tie between the two orderings xRy and yRx which we have written as {xRy, yRx}. However, Arrow quickly defines xRy and yRx as xIy (Definition 2, p. 14). The crux of the matter is: what does "and" mean? If "and" is the "logical and" which we denote AND, then "xRy AND yRx implies xIy" is a reasonable definition, but to assert that both the events xRy and yRx may occur in accordance with Axiom I is not the same as asserting xRy AND yRx. We use the lower case "and" to indicate the English connective as opposed to the "logical and", and, if both events (xRy and yRx) occur we denote this as {xRy, yRx} and call it a tie. One could then assign the domain case, **N(x,y)** = **N(y,x)**, to xIy instead of to {xRy, yRx} except for the fact that to be correct, if the domain uses only the operator R, then the range solutions should use only the operator R and not an operator defined in terms of R. This is probably the reason that Arrow violates the Principle of Neutrality in his analysis of binary majority rule as we will see later: he would have to assign the tie case, **N(x,y)** = **N(y,x)**, to xIy. Elsewhere, Arrow's presentation continues to assume that a tie in the domain implies a social indifference.

This is only *one* interpretation of a tie which accords with Majority Rule (PI2) as given above. It is not *necessary* to define a tie as an indifference. This is an arbitrary construction by Arrow which obviates the necessity for considering a more general approach to ties. Arrow simply *assumes* Majority Rule (PI2) with no further discussion, but this is not the most general way to proceed. There is nothing rational or ethical about approaching the issue of social choice in this manner as opposed to a more general treatment of the situation. Arrow is clearly taking a tie and *defining* it as an indifference.

Arrow's statement that in a "strong ordering ... no ties are possible" violates the common sense notion considered above in which (when only preferences are considered) n/2 voters prefer a to b and n/2 voters prefer b to a. Clearly, this is a tie, and clearly we *cannot* have the social choice aIb since the indifference operator is not a part of the domain or the range. The social choice must be {aPb, bPa} or some other representation of a tie. However, Arrow's language, [in a] "strong ordering ... no ties are possible" makes it clear that to Arrow a tie *is the same as* an indifference so that if indifferences are not possible (as they wouldn't be in a strong ordering), then *ties* are not possible. This is clearly incorrect. Ties are possible even when indifferences are not possible. Arrow's arbitrary narrowing of the analysis limits his result to a negative (for m > 2) in regards to the existence of SWFs which meet his criteria.

An important thing to keep in mind here is that a tie refers to elements of the range and not to alternatives. If there are just two alternatives in an election, we say, sloppily, that there is a possibility of a tie between x and y when what we mean (considering just preference relationships) is that there is a possibility of a tie between xPy and yPx which are the social choices. In other words, xPy and yPx are the social choices for which a tie may exist not x and y which are the alternatives. The same should hold true for xRy and yRx. Since the SWF produces orderings, the ties which should be considered are ties between *orderings* — not ties between *alternatives*.

Arrow's Treatment of Majority Rule

Arrow's proof that social choice is possible for two alternatives is dependent on the way he defines Citizens' Sovereignty. If the stronger Principle of Neutrality is assumed, Arrow's proof falls apart due to his treatment of the tie case. Arrow's definition of Majority Rule is as follows:

MAJORITY RULE (A) "DEFINITION 9: *By the method of* majority decision *is meant the social welfare function in which xRy holds if and only if the number of individuals such that* xR_iy *is at least as great as the number of individuals such that* $yR_i\,x$*."*

Therefore, the case in which N(x,y) = N(y,x) would be decided xRy. But this clearly violates the notion of treating both alternatives in the same manner. Murakami (1968) states: "As long as we are considering the world of two alternatives, self-duality can be regarded as impartiality or neutrality with respect to alternatives. A self-dual social decision function has exactly the same structure regarding issue x against y as it does regarding issue y against x." Therefore, Arrow's proof gets by only by weakening the Principle of Neutrality to the Condition of Citizens' Sovereignty. One would think that, since Arrow provided for the possibility of the tie set, xRy and yRx, in Axiom I, it should be called for in this case. Perhaps the reason Arrow does not consider the tie case is that he has defined xRy AND yRx as xIy, and this would result in a particular case being outside the R system; indeed, the SWF would involve I which has been derived from R by virtue of a definition. Also, Arrow's 5 criteria are, in many respects, the weakest forms of those criteria which is appropriate for the proof that no SWF exists which meet those criteria. However, for m = 2, Arrow proves that a SWF does in fact exist, and, therefore, it would seem more appropriate to strengthen the criteria when proving existence.

In showing connectivity Arrow states: “Clearly, always either $N(x,y) \geq N(y,x)$ or $N(y,x) \geq N(x,y)$, so that, for all x and y, xRy or yRx ... and R is connected.” This is an incorrect statement. One could say correctly that ‘either $N(x,y) \geq N(y,x)$ or $N(y,x) > N(x,y)$’ or ‘either $N(x,y) > N(y,x)$ or $N(y,x) \geq N(x,y)$’ or ‘either $N(x,y) > N(y,x)$ or $N(y,x) > N(x,y)$ or $N(y,x) = N(x,y)$.’ The latter restatement then would suggest the conclusion that either xRy or yRx or {xRy, yRx}. However, Arrow’s definition of majority rule would have to be changed to allow for the tie case if both alternatives are to be treated similarly. With these changes one could then go on to prove that social choice that upholds the Principle of Neutrality is indeed possible for the case of two alternatives *but only by acknowledging the tie situation*. Arrow’s analysis and proof that a SWF exists for $m = 2$ is dependent on the fact of treating the two alternatives dissimilarly or, looked at from a different perspective, the proof that social choice is possible for two alternatives is dependent on having weakened the Principle of Neutrality to the Condition of Citizens’ Sovereignty.

In his proof, Arrow goes on to show that R is transitive. However transitivity only applies when there are three or more alternatives. Showing a relationship is transitive when there are only two alternatives involved is unnecessary to say the least.

Arrow goes on: “Now consider Condition 2. Let $R_1, \ldots, R_n$ be such that xPy, i.e., xRy and not yRx.” According to Definition 9, however, this is *never* true. The social ordering or range element produced by the SWF is *always* either xRy or yRx. xPy is not an acceptable range element but is derived from one, i.e., xPy is defined to be not yRx. It can never be assigned as a solution by the SWF. Arrow makes the mistake here of assuming that, if $N(x,y) > N(y,x)$, xPy which is not true according to his definition. What is true is the following: if $N(x,y) > N(y,x)$, xRy. There is no domain element, $R_1, \ldots, R_n$, such that xPy.

The problem here goes back to the specification of Arrow's Condition 2, the Positive Association of Social and Individual Values:

"Let R_1, ..., R_n and R'_1, ..., R'_n be two sets of individual ordering relations, R and R' the corresponding social orderings, and P and P' the corresponding social preference relations. Suppose that for each i the two individual ordering relations are connected in the following ways: for x' and y' distinct from a given alternative x, x' R'_i y' if and only if x' R_i y'; for all y', xR_i y' implies xR'_i y'; for all y', xP_i y' implies xP'_i y'. Then, if xPy, xP'y."

This Condition contains many lapses of logic which are enumerated as follows:

1) The system admits R_i data which constitute the domain and converts this to R data which constitutes the range. The condition should be couched in terms of R and R_i only, i.e., in terms of operators within the system.

2) P_i and P'_i are meaningless since each individual specifies R_i, preference information has been abstracted from and hence is unknown.

23 P and P' are meaningless since the social choice must be R or R' according to Definition 9.

Condition 2 should be stated as follows:

Condition 2*: Let R_1, ..., R_n and R'_1, ..., R'_n be two sets of individual ordering relations and R and R' the corresponding social orderings. Suppose that for each i the two individual ordering relations are connected in the following ways: for x' and y' distinct from a given alternative x, x' R'_i y' if and only if x' R_i y' and for all y', xR_i y' implies xR'_i y'. Then xRy implies xR'y.

Heuristically, paraphrasing Arrow, we have two sets of orderings. In the initial position, the orderings are given by R_1, ..., R_n. This gives rise to a social ordering in which xRy. Quoting Arrow: “Suppose values subsequently change in such a way that for each individual the only change in relative rankings, if any, is that x is higher in the scale than before.” Paraphrasing again, if we call the new ordering $R'_1, ..., R'_n$ and the corresponding social ordering R', then we would certainly expect that xR'y. If, however, the initial social ordering was yRx, then the second ordering could also be xR'y.

We quote Arrow’s heuristic language and then compare it to our own in order to delineate the difference and point out where Arrow logically went astray.

Arrow: “The condition that x be not lower on the R_i' scale than x was on the R_i scale means that x is preferred on the R_i' scale to any alternative to which it was preferred on the old (R_i) scale and that also x is preferred or indifferent to any alternative to which it was formerly indifferent. The two conditions of the last sentence, taken together, are equivalent to the following two conditions: (1) x is preferred on the new scale to any alternative to which it was formerly preferred; (2) x is preferred or indifferent on the new scale to any alternative to which it was formerly preferred or indifferent. In symbols, for all y', xR_iy' implies $xR_i'y'$ and xP_iy' implies xP'_iy'.”

Our language: The condition that x be not lower on the R_i' scale than x was on the R_i scale means that x is preferred or indifferent on the R'_i scale to any alternative to which it was preferred or indifferent on the R_i scale and that it *may be* preferred or indifferent on the R_i' scale to any alternative which was preferred or indifferent to it on the R_i scale. In symbols, for all y', xR_iy' implies $xR_i'y'$.

Arrow goes on to say in his proof: “As for Condition 4 for any x and y, suppose that individual orderings were such that yP_ix for all i.” Individual orderings can never be such that yP_ix for all i. Individual orderings can only be of the form xR_iy or yR_ix by definition. Arrow again confuses the case N(y,x) > N(x,y) which yields yRx according to Definition 9 with yielding yPx.

The same confusion ensues when it comes to proving the SWF is not dictatorial. There is no valid reason why in Arrow's Definition 6, P should be used at all. Arrow's Definition 6 is the following:

"A social welfare function is said to be dictatorial *if there exists an individual i such that, for all x and y,* xP_iy *implies xPy regardless of the orderings* $R_1, ..., R_n$ *of all individuals other than i, where P is the social preference relation corresponding to* $R_1, ..., R_n$*."*

We don't know if xP_iy since the domain is specified in terms of R_i for $1 \leq i \leq n$. Therefore, xP_iy and xPy have no meaning as individual or social orderings. The Definition in order to be valid in the R-system must be stated as follows:

Definition 6': A social welfare function is said to be dictatorial if there exists an individual i such that, for all x and y, xR_iy implies xRy regardless of the orderings $R_1, ..., R_n$ of all individuals other than i, where R is the social preference relation corresponding to $R_1, ..., R_n$.

With these changes Arrow's proof starts to make sense.

Majority Rule (R) can be proved along the lines of Majority Rule (P). Majority Rule as stated by Arrow's Definition 9 can also probably be proved so long as the Condition of Citizen's sovereignty is retained. However, Arrow's version of majority rule could not be proved if the Principle of Neutrality were invoked and ties were not considered whereas Majority Rule (R) can be proved with Neutrality if ties are taken into account.

Conclusions

The discussion in this paper has been confined to the binary case of two alternatives or issues. It has been shown that various forms of Majority Rule are possible for the cases where the relationship operator is P, strict preference; P and I, preference and indifference; and R, preference or indifference. All of these forms of Majority Rule are SWFs in that they meet Arrow's Conditions and Axioms. In particular they meet a strengthened version of Arrow's Condition 4, Citizens' Sovereignty known as Neutrality with regard to which all alternatives must be treated in the same manner.

Arrow's treatment of the binary case, in which he proves that a SWF exists, violates the Principle of Neutrality in that it treats the case in which the number of voters that specify xR_iy is equal to the number of voters who specify yR_ix differently with respect to the alternatives x and y. In particular, it assigns the solution to xRy when the votes are tied. This is acceptable within the terms of Citizens' Sovereignty but not within the terms of Neutrality. We see that Arrow's Conditions themselves, far from being cast in concrete as rational and ethical, are somewhat arbitrary and unethical. If Arrow had assumed Neutrality instead of Citizens' Sovereignty, he would not have been able to prove that social choice for m = 2 is possible unless he had treated the tie case properly.

Arrow's proof that social choice is possible for m = 2 is fraught with mistakes. In particular, if he is operating within the R-system, i.e., individual and social orderings are expressed in terms of the R operator, then his conditions and proofs must be dealt with in terms of the R system. He repeatedly assumes an individual or social knowledge of P information which doesn't exist.

Proper treatment of the tie case also opens the door for the existence of SWFs which are rational and ethical for m > 2 (Lawrence, 1998). Consequently ties of the form {xQyQz, yQzQx, zQyQx} are possible for the case m = 3, where Q is chosen from the set {P,I}. In fact the rational and ethical conditions can be strengthened considerably, and it can be proven that SWFs exist for any value of m, the number of alternatives and n, the number of voters. (Lawrence, 1998).

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