The Possibility of Social Choice for 3 Alternatives

by

John Clifton Lawrence
General Algorithm
POB 230351
Encinitas, CA 92023
Phone/fax: 760-633-3778
Web site: http://www.genalg.com
E-mail: jlawrence@genalg.com

September 15, 1998

Revised: 2006

PO Box 126155
San Diego, CA 92112

Email: j.c.lawrence@cox.net
Web Site: http://www.socialchoiceandbeyond.com
Blog: http://willblogforfood.typepad.com

Abstract

In "Social Choice and Individual Values" Arrow (1951) discusses the possibility of ties for binary orderings. In particular, either xRy, yRx or both xRy and yRx are possible solutions when m (the number of alternatives) = 2 where R is the social "preference or indifference" operator. This is demanded by the axiom of completeness. Let us write "both xRy and yRx" as {xRy, yRx}. In the preceding sentence the word "and" is the English connective as distinguished from the logical and which we write "AND. " If half the voter/consumers have xR_iy and half have yR_ix (where R_i is the individual "preference or indifference" operator), it would be natural to assume (as one possibility) that the social ordering is {xRy, yRx} which we define as a tie. By extension, for three alternatives, if half the voter/consumers have xP_iyP_iz and half have yP_ixP_iz, it would be natural to assume (as one possibility) that the social ordering is the tie {xPyPz, yPxPz}. This reduces correctly to the binary solutions {xPy, yPx}, xPz and yPz when the appropriate alternative is removed both at the individual and the social levels. Arrow only considers ties among *alternatives* via his social choice function, C(S), and not ties among *orderings*. Since he demands orderings as the solutions for a Social Welfare Function (SWF), it would be more natural to consider ties among orderings which are also demanded by the axiom of completeness. Considerations of ties among orderings leads to the possibility of legitimate SWFs which are presented for m = 3 and which comply with the axioms of connectivity and transitivity and a strengthened version of Arrow's 5 criteria.

Key Words: social choice, Arrow, Condorcet, voting, paradox of voting, algorithm, social welfare function, impossibility theorem, general possibility theorem

List of Symbols

R_i

R

aR_ib

aRb

P_i

P

aP_ib

aPb

I_i

I

aI_ib

aIb

Introduction

In this paper, our goal is not to make an incremental contribution to the field of social choice, but to make a revolutionary one. This paper is not about offering a workaround for Arrow's Impossibility Theorem, but is about overturning it. In so doing we accept all of Arrow's framework except that part which is counterintuitive and flawed. We show that it is possible to have a Social Welfare Function (SWF) which complies with Arrow's criteria. The flawed part of Arrow's theory has to do with his treatment of ties. Arrow only considers ties among *alternatives,* but the range of the SWF consists of *orderings*. In a real sense, instead of voting for individual candidates or alternatives, the voters are voting for orderings. Therefore, ties among orderings must be considered. When they are, SWFs are possible and demonstrable (Lawrence, 1998).

We assume alternatives of the form a, b, c ... x, y, z; a preference relationship, P, an indifference relationship, I and a "preference or indifference" relationship, R. We assume a set of voter/consumers each of whom has an ordering over a given set of m alternatives characterized by a list of preferences; or preferences and indifferences. For example, $aP_ib\ P_ic$ would characterize a preference ordering over an alternative set of three alternatives by the i^{th} voter/consumer. $aI_ib\ P_ic$ would characterize a preference and indifference ordering over the set.

We assume a SWF which is a mapping from the set of individual orderings to a social ordering. P, I and R without subscripts represent social orderings. Therefore, a social ordering would be of the form aPbPc or aPbIc or a set of consistent binary orderings of the form {aRb, bRa, aRc, cRa, bRc, cRa}, for example. An expresssion of the form aRbRc is meaningless since we need to know both aRb and bRa, for example, to maintain a 1-1 relationship between P and I information and R information. We assume a universal SWF which means that there is a mapping from every possible combination of individual orderings (the domain) to one of the set of every possible social ordering (the range). Each element of the range represents a potential social ordering. Therefore, a social ordering is assigned to each domain point by the SWF.

Arrow's (1951, p. 13) Axiom 1 states: "For all x and y, either xRy or yRx." A relation R which satisfies Axiom 1 is said to be complete and reflexive since xRx. The "or" in the definition is the inclusive or so that "the word 'or' in the statement of Axiom 1 does not exclude the possibility of both xRy and yRx. " We write "both xRy and yRx" as {xRy, yRx}.

The choice function, C(S), is defined by Arrow (p. 15) as follows: "C(S) is the set of all alternatives in S such that, for every y in S, sRy. " As such it can be used to specify ties among *alternatives* if the set, C(S), contains more than one element. Sen (1970, p. 48) says, "Arrow's impossibility theorem is precisely a result of demanding social orderings as opposed to choice functions." In other words, if the solutions required were simply alternatives, Arrow's Impossibility Theorem would not apply. Since Arrow only uses it in his specification of the Condition of the Independence of Irrelevant Alternatives, it would have been more natural (and certainly stronger) to define C(S) as an *ordering* over a subset instead of the highest ranking alternative or set of alternatives in a subset. We define an ordering function herein which strengthens the Condition of Independence of Irrelevant Alternatives and allows for ties among orderings as well as ties among alternatives.

The three possible range assignments according to Arrow's Axiom 1 are xRy, yRx, {xRy, yRx}. Arrow goes on to define xRy AND yRx as xIy and not yRx as xPy. Arrow assumes a knowledge of P and I. Therefore, he should have defined R in terms of P and I instead of the other way round. From the point of view of this paper, we will demonstrate our results in terms of P and I rather than R.

In the P and I world Axiom 1 can be restated as "For all x and y, either xPy, yPx, xIy, {xPy,yPx}, {xPy, xIy}, {yPx, xIy} or {xPy, yPx, xIy}" {xPy, yPx} might (but does not necessarily have to) be the social ordering if half the voter/consumers prefer x to y and half prefer y to x. Similarly, {xPy, xIy} might be the social ordering if half the voter/consumers prefer x to y, and half are indifferent between x and y. Finally, {xPy, yPx, xIy} might be the social ordering if a third prefer x to y, a third prefer y to x and a third are indifferent between y and x. If P and I are primary and R defined in terms of them, then Axiom 1 can be restated as the following: "For all x and y, either not yRx, not xRy, xRy AND yRx, {not yRx,not xRy}, {not yRx, xRy AND yRx }, {not xRy, xRy AND yRx } or {not yRx, not xRy, xRy AND yRx }." Sen (1970, p. 41-46) manages to reconstruct essentially the same proof using P and I and without using R at all.

The Binary Case

We take as an example the binary case of two alternatives, x and y, and n voters. This is the typical, traditional voting situation and we assume the majority voting rule. The individual voters specify either xP_iy or yP_ix. We will not include indifferences for now. The corresponding social orderings are xPy and yPx. If n is an even number and n/2 voters specify xP_iy while the other n/2 voters specify yP_ix, then we clearly have a tie which we indicate {xPy, yPx}. Note that P does not have to be reflexive for this voting rule to be perfectly rational, but it does have to be complete $^{i.e.}$ either xPy, yPx or {xPy, yPx}. These then comprise the set of range elements that can be considered social orderings. Heuristically and intuitively, we *must* provide for the possibility of a tie as a valid range option. We know from experience that such an outcome is possible. Yet Arrow does not consider ties in his discussion of the binary case which we will discuss further below.

If we consider both P and I, then the individual voter/consumers specify xP_iy, yP_ix or $xI_iy = yI_ix$. The corresponding social orderings are xPy, yPx, xIy, {xPy, yPx}, {xPy, xIy}, {yPx, xIy}, {xPy, yPx, xIy}. xIy might heuristically be appropriate if the majority of voters are indifferent between x and y but not appropriate if half the voters prefer x to y and half, y to x. For the sake of completeness, both solutions are available. Note that the domain in the P and I world *includes* the domain of the P world. Since I is reflexive and P is not reflexive, some of the individual orderings are reflexive, namely xI_iy and only one social ordering is reflexive: xIy. Since various writers (notably Sen) have proved Arrow's Impossibility Theorem without using R (which is reflexive) and only using P and I (which taken together aren't), we conclude that reflexivity is not necessary to the analysis and certainly not necessary for rationality.

Let N(x,y) be the number of voters who vote xP_iy , and N(y,x) be the number who vote yP_ix. The majority rule which we assumed above connecting domain and range elements can be more formally stated as follows: If N(x,y) > N(y,x), the social ordering is xPy. If N(y,x) > N(x,y), the social ordering is yPx. If N(x,y) = N(y,x) (which can only happen if n is even), the social ordering is a tie {xPy, yPx}.

Arrow (1951, p. 13-14) claims to treat ties. He asserts: "...Axioms I and II do not exclude the possibility that for some distinct [alternatives] x and y, both xRy and yRx. A strong ordering, on the other hand, is a ranking in which no ties are possible." This is not correct. Clearly, ties are possible for the strong ordering P as discussed above. Arrow is implying here that a social ordering could consist of the tie set {xRy, yRx}, but he assumes that "both xRy and yRx" is equivalent to xRy AND yRx which is the same as xIy. However, this is not true in general but only if one defines it this way.

Arrow's proof that social choice is possible for two alternatives is questionable because he doesn't deal with the tie case, N(x,y) = N(y,x), rigorously. Arrow (1951, p. 46) states: "DEFINITION 9: *By the method of* majority decision *is meant the social welfare function in which xRy holds if and only if the number of individuals such that* $xR_i y$ *is at least as great as the number of individuals such that* $yR_i x$*."*

Therefore, the case in which N(x,y) = N(y,x) would be decided xRy. But this violates the principal of neutrality or self-duality that requires every alternative to be treated in exactly the same manner. Murakami (1968, p. 33) states: "As long as we are considering the world of two alternatives, self-duality can be regarded as impartiality or neutrality with respect to alternatives. A self-dual social decision function has exactly the same structure regarding issue x against y as it does regarding issue y against x." Self-duality is a stronger version of Arrow's Condition 3 — Citizen's Sovereignty, but one would think that, since Arrow provided for the possibility of the tie set, {xRy, yRx}, in Axiom I, it should be called for in this case. There is no reason to prefer x over y in this situation by calling for xRy as the solution in the tie case as opposed to yRx. You can't have it both ways. If you aren't going to allow the existence of tie sets as legitimate social choices, then there is no legitimate social choice in the binary case either since you would have to assign the case N(x,y) = N(y,x) to either xRy or yRx which violates neutrality. On the other hand, if tie sets are acceptable, then they must be admitted as potential solutions for cases such that m > 2, and this opens the door for legitimate social orderings which contradict Arrow's Impossibility Theorem.

In showing connectivity Arrow states: "Clearly, always either N(x,y) ≥ N(y,x) or N(y,x) ≥ N(x,y), so that, for all x and y, xRy or yRx." This is an incorrect statement. One could say correctly that 'either N(x,y) ≥ N(y,x) or N(y,x) > N(x,y)'; or 'either N(x,y) > N(y,x) or N(y,x) ≥ N(x,y)'; or 'either N(x,y) > N(y,x) or N(y,x) > N(x,y) or N(y,x) = N(x,y).' The latter restatement then would suggest the conclusion that either xRy or yRx or {xRy, yRx} which would be consistent with Axiom 1. However, Arrow's definition of majority rule would have to be changed to allow for the tie case. With these changes one could then go on to prove that a social ordering is indeed possible for the case of two alternatives, but not allowing the acceptance of the tie case leads to the conclusion that a social ordering is impossible for the binary case as well. It is also counterintuitive to reality!

But Definition 9 has other problems. Let's say half the voters have xI_iy and half have yP_ix. Then according to Definition 9, xRy. But this is ridiculous!

The Ternary Case — n odd

When the "inclusive or" interpretation of Axiom I is extended to three alternatives, we would have social ordering solutions, for instance, of the form $\{aQ^1bQ^2c,\ bQ^3aQ^4c,\ cQ^5aQ^6b\}$, where Q^j is chosen from the set {P,I}. For example, let us imagine a situation in which there are 3 alternatives and 6 voter/consumers and we exclude I as an operator. There are 6 possible individual orderings: aP_ibP_ic, aP_icP_ib, bP_iaP_ic, bP_icP_ia, cP_iaP_ib, cP_ibP_ia. Let us assume that each of the 6 voter/consumers specifies a different ordering from among the above six possible orderings. The intuitive and heuristic solution is a tie among all the possible orderings. Similarly, there are 24 possible orderings for 4 alternatives, and, for the case of 24 voter/consumers, each specifying a different ordering, common sense would dictate a tie among all the possible social orderings. A similar case can be made for m=5, 6, ... These are the broadest conceivable tie sets, and will be called maximal tie sets. Tie sets involving less than the total number of orderings are also possible and demanded by the completeness requirement.

An important thing to keep in mind here is that a tie refers to orderings and not to alternatives. The choice function C(S) would specify a tie between the alternatives x and y if xIyRz were the social ordering, for example. We are considering here ties among the orderings themselves and not just among the "top slot" of those orderings.

We now proceed to demonstrate solutions which are social orderings for a specific SWF for the case m = 3 which satisfy a strengthened version of Arrow's conditions. Let us assume alternatives x, y and z and n (odd) voter/consumers. We exclude the indifference operator for now so that each voter must vote xP_iy or yP_ix. As a consequence of Arrow's Condition 3, the Independence of Irrelevant Alternatives, we know that "knowing the social choices made in pairwise comparisons determines the entire social ordering." Accordingly, we consider the social choices of the alternatives two by two. Our SWF is as follows.

If N(x,y) > N(y,x), then xPy. If N(y,x) > N(x,y), then yPx. At the ternary level we have 8 cases:

Case 1:	xPy, xPz, yPz
Case 2:	xPy, xPz, zPy
Case 3:	xPy, zPx, yPz
Case 4:	xPy, zPx, zPy
Case 5:	yPx, xPz, yPz
Case 6:	yPx, xPz, zPy
Case 7:	yPx, zPx, yPz
Case 8:	yPx, zPx, zPy

According to the Condorcet (1785) method for determining the outcome of an election, we consider each of the alternatives in pairs, determine the winner for each pair and then determine the final social ordering by combining these results. We use the Condorcet method in our SWF for the above cases in which it actually produces a result. Therefore, we have the following:

Case	Social Ordering
1	xPyPz
2	xPzPy
4	zPxPy
5	yPxPz
7	yPzPx
8	zPyPx

This leaves only cases 3 and 6. Consider the solution {xPyPz, yPzPx, zPxPy} for Case 3. We call a reduced ordering or reduced solution an ordering with one or more alternatives removed. If we consider {xPyPz, yPzPx, zPxPy} and remove z, we get {xPy, yPx, xPy}. Combining terms we have {2xPy, yPx}. If we choose the most numerous of xPy and yPx as the solution, we get xPy by 2 to 1 which we know to be true.

Likewise, if we reduce {xPyPz, yPzPx, zPxPy} by y, we get {xPz, zPx, zPx} or {xPz, 2zPx}. 2zPx > xPz and we take zPx as the reduced solution which agrees with the known binary solution. Similarly, if we remove x from the social solution, we have {yPz, yPz, zPy} which yields yPz. Accordingly, our SWF algorithm is as follows:

> 1) Choose the Condorcet solution if it exists.
> 2) If the Condorcet solution doesn't exist, construct a solution such that, when the solution is reduced by any single alternative, the most numerous of the remaining binary relationships is the same as the binary solution.

Notice that our algorithm will always produce consistent results if the ternary solution is generated from the binary solution in such a way that there is a 2 to 1 ratio between the correct binary solution and the incorrect binary solution and then we take the larger of the two as our reduced solution. We construct our solutions in this manner in order to be compliant with Arrow's Condition 3, the Independence of Irrelevant Alternatives. Satisfying the other Conditions is then trivial as we shall show. Whether or not such a solution always exists will be answered affirmatively elsewhere. Here all we need to show is the existence of a solution for Case 6. Consider the solution
{yPxPz, xPzPy, zPyPx}. Reduction by z yields yPx; by y, xPz; by x, zPy which agrees with the known binary case and is consistent with the above definition.

Therefore, we have demonstrated a consistent algorithm for the SWF which yields the same social orderings when reduced from the ternary case to the binary case as those produced at the binary level directly from the domain. There is complete consistency of social orderings and not just of alternatives produced by the choice function. The choice function only produces the top position in an ordering. We demand consistency over all orderings which can be produced by reducing a social ordering and this strengthens Arrow's Condition 3.

Arrow (1951, p. 26) states that "...suppose that an election system has been devised whereby each individual lists all the candidates in order of his preference and then, by a preassigned procedure, the winning candidate is derived from these lists. ...Suppose an election is held, with a certain number of candidates in the field, each individual filing his list of preferences , and then one of the candidates dies. Surely the social choice should be made by taking each of the individual's preference lists, blotting out completely the dead candidate's name, and considering only the orderings of the remaining names in going through the procedure of determining a winner.” This is precisely what we have done in choosing our SWF. Notice that it is completely consistent with the solutions for those cases determined by the Condorcet method.

Arrow’s Condition 3 is the following:

“Let $R_1,..., R_n$ and $R'_1,..., R'_n$ be two sets of individual orderings and let C(S) and C'(S) be the corresponding social choice functions. If, for all individuals i and all x and y in a given environment S, xR_iy if and only if $xR'_i\,y$, then C(S) and C'(S) are the same (independence of irrelevant alternatives).”

Notice that Arrow only requires consistency in the top position of the orderings over S. Therefore, assuming Condition 3, if S = {x,y,z}, R = wQxQyQzQa and R' = aQxQzQyQw, then C(S) = C'(S) = x and Arrow's Condition 3 is satisfied although the social orderings R and R' have different orderings over the set S. We require the entire social orderings over the set S to be identical as well thus strengthening Arrow's Condition 3.

Let O(S) be the social ordering function over a set of alternatives S. O(S)=U where U is the social ordering. Then, let V be the social ordering over a set of alternatives T with $S \subset T$. O(T)=V. Then, xRy in U iff xRy in V.

Now we can restate Condition 3 as the following:

“Let $R_1,..., R_n$ and $R'_1,..., R'_n$ be two sets of individual orderings and let O(S) and O'(S) be the corresponding social ordering functions. If, for all individuals i and all x and y in a given environment S, xR_iy if and only if $xR'_i\,y$, then O(S) and O'(S) are the same (independence of irrelevant alternatives).”

The Ternary Case — n even

When n is even we have a total of 27 cases. We have already considered the first 8 cases above. For convenience we define {xPy, yPx} as xTy. In addition there is one more tie possibility, a three way tie: N(x,y) = N(y,x) = N(y,z) = N(z,y) = N(x,z) = N(z,x). We write this as {xPy, yPx, yPz, zPy, xPz, zPx} and define this as xTyTz. Solutions for the remaining cases are shown below.

Case	Binary Solutions	Ternary Solution
9:	xPy, xPz, yTz	xPyTz
10:	xPy, zPx, yTz	{zPxPy, xPyTz, yTzPx}
11:	yPx, xPz, yTz	{yPxPz, xPyTz, yTzPx}
12:	yPx, zPx, yTz	yTzPx
13:	xPy, xTz, yPz	{xPyPz, yPxTz, xTzPy}
14:	xPy, xTz, zPy	xTzPy
15:	yPx, xTz, yPz	yPxTz
16:	yPx, xTz, zPy	{zPyPx, yPxTz, xTzPy}
17:	xTy, xPz, yPz	xTyPz
18:	xTy, xPz, zPy	{xPzPy, xTyPz, zPxTy}
19:	xTy, zPx, yPz	{yPzPx, xTyPz, zPxTy}
20:	xTy, zPx, zPy	zPxTy
21:	xPy, xTz, yTz	{xPyTz, xTzPy, xTyTz}
22:	yPx, xTz, yTz	{yPxTz, yTzPx, xTyTz}
23:	xTy, xPz, yTz	{xPyTz, xTyPz, xTyTz}
24:	xTy, zPx, yTz	{zPxTy, yTzRx, xTyTz}
25:	xTy, xTz, yPz	{yPxTz, xTyPz, xTyTz}
26:	xTy, xTz, zPy	{zPxTy, xTzRy, xTyTz}
27:	xTy, xTz, yTz	xTyTz

The P and I World

In the P and I world, we have two relationships to deal with. Let's just consider 2 alternatives for now. Let N(x,y) be the number of individual voter/consumers who prefer x to y, and M(x,y) be the number who are indifferent between x and y. There are then 13 possibilities as follows:

Case 1: N(x,y) > N(y,x) > M(x,y)

Case 2:	N(y,x) > N(x,y) > M(x,y)
Case 3:	N(x,y) > M(x,y) > N(y,x)
Case 4:	N(y,x) > M(x,y) > N(x,y)
Case 5:	M(x,y) > N(x,y) > N(y,x)
Case 6:	M(x,y) > N(y,x) > N(x,y)
Case 7:	N(x,y) > N(y,x) = M(x,y)
Case 8:	N(y,x) > N(x,y) = M(x,y)
Case 9:	M(x,y) > N(x,y) = N(y,x)
Case 10:	N(x,y) = N(y,x) > M(x,y)
Case 11:	M(x,y) = N(x,y) > N(y,x)
Case 12:	M(x,y) = N(y,x) > N(x,y)
Case 13:	M(x,y) = N(x,y) = N(y,x)

One possible binary decision rule might the following. If N(x,y) > N(y,x) and M(x,y), then xPy. If N(y,x) > N(x,y) and M(x,y), then yPx. If M(x,y) > N(x,y) and N(y,x), then xIy. If N(x,y) = N(y,x) > M(x,y), then {xPy, yPx}=xTy. If N(x,y) = M(x,y) > N(y,x), then {xPy, xIy}. If N(y,x) = M(x,y) > N(x,y), then {yPx, xIy}. If N(x,y) = N(y,x) = M(x,y), then {xPy, yPx, xIy}. There would be 7 possible social orderings at the binary level. At the ternary level would be 7^3 possible combinations each of which would require a social ordering.

However, the SWF need not make use of every possible range element in providing a mapping from domain to range. We only need to make sure that there is at least one set of connections which satisfy Arrow's criteria and axioms. Accordingly, we only consider the following binary social orderings: xPy, yPx, xTy = {xPy, yPx}, xIy, and the following binary decision rule.

		Social Ordering
Case 1:	N(x,y) > N(y,x) > M(x,y)	xPy

Case 2:	$N(y,x) > N(x,y) > M(x,y)$	yPx
Case 3:	$N(x,y) > M(x,y) > N(y,x)$	xPy
Case 4:	$N(y,x) > M(x,y) > N(x,y)$	yPx
Case 5:	$M(x,y) > N(x,y) > N(y,x)$	xIy
Case 6:	$M(x,y) > N(y,x) > N(x,y)$	xIy
Case 7:	$N(x,y) > N(y,x) = M(x,y)$	xPy
Case 8:	$N(y,x) > N(x,y) = M(x,y)$	yPx
Case 9:	$M(x,y) > N(x,y) = N(y,x)$	xIy
Case 10:	$N(x,y) = N(y,x) > M(x,y)$	xTy
Case 11:	$M(x,y) = N(x,y) > N(y,x)$	xPy
Case 12:	$M(x,y) = N(y,x) > N(x,y)$	yPx
Case 13:	$M(x,y) = N(x,y) = N(y,x)$	xTy

At the ternary level we have $64 = 4^3$ cases to consider as follows. We present the solutions in Appendix 1.

THEOREM For m = 3 and any n there exists a SWF relative to the relations P and I for which the social orderings consist of either unique rankings or of ties of at most three orderings.

PROOF By inspection.

Proof that Algorithm Satisfies Arrow's Axioms and Criteria

Axiom I: **Connectivity**

Either xPy, yPx, xIy or {xPy, yPx} and xIx.

Axiom II: **Transitivity**

xPy AND yPz imply xPz; xPy AND yIz imply xPz; xIy AND yPz imply xPz; xIy AND yIz imply xIz; xPy AND yTz imply xPz; xTy AND yPz imply xPz; xTy AND yTz imply xTz; xIy AND yTz imply xIz; xTy AND yIz imply xIz.

Condition 1: **Existence of a free triple**

Arrow only required that some set of three alternatives be available for any logical ordering. Our algorithm assigns solutions for every logical ordering of every individual voter.

Condition 2: **Positive Association of Individual and Social Values**

This Condition requires that, if every individual voter raises some candidate in his "preference or indifferenve" list, that candidate must not be lowered in the social choice. The algorithm considered here satisfies an even stronger criterion which is, if any individual voter raises a candidate in his "preference or indifference" list, the "preference or indifference" lists of all other voters remaining the same, then that candidate must not be lowered in the social choice.

Since the social choice is based on the choices made on binary pairs, let us consider only one voter, voter i, and only two candidates, x and y. Let us say voter i originally had xP_iy and then switched to yP_ix. Let us assume that, originally, xPy. If the majority of voter/consumers still have xP_jy after voter i's change, then society will still have xPy; if there is now a majority for y over x, society will have yPx. However, there is the possibility that the change of one vote will change the social ordering to xTy. If, however, originally xTy, then, after voter i's change, society will have yPx. Therefore, xPy or xTy can change to xTy or yPx.

At stage 3, if we originally had xPy, then the social ordering would have to be one of the following: xPyPz, xPzPy, zPxPy, xPyTz, xTzPy, zTxPy, {xPyPz, yPzPx, zPxPy}, {zPxPy, xPyTz, yTzPx}, {xPyPz, yPxTz, xTzPy}, {xPyTz, xTzPy, xTyTz}. If we originally had xTy, then the social ordering would have to be one of the following: xTyPz, {xPzPy, xTyPz, zPxTy}, {yPzPx, xTyPz, zPxTy}, zPxTy, {xPyTz, xTyPz, xTyTz}, {zPxTy, yTzPx, xTyTz}, {yPxTz, xTyPz, xTyTz}, {zPxTy, xTzPy, xTyTz}, xTyTz.

After voter i's change the possible solutions if there is a social change are the following: xPyPz → yPxPz or xTyPz, xPzPy → {yPxPz, xPzPy, zPyPx} or {xPzPy, xTyPz, zPxTy}, zPxPy → zPyPx or zPxTy, xPyTz → {yPxPz, xPyTz, yTzPx} or {xPyTz, xTyPz, xTyTz}, xTzPy → {zPyPx, yPxTz, xTzPy or {zPxTy, xTzPy, xTyTz}, zTxPy → {zPyPx, yPxTz, xTzPy} or {zPxTy, xTzPy, xTyTz}, {xPyPz, yPzPx, zPxPy} → yPzPx or {yPzPx, xTyPz, zPxTy}, {zPxPy, xPyTz, yTzPx} → yTzPx or {zPxTy, yTzPx, xTyTz}, {xPyPz, yPxTz, xTzPy} → yPxTz or {yPxTz, xTyPz, xTyTz}, {xPyTz, xTzPy, xTyTz} → {yPxTz, yTzPx, xTyTz} or xTyTz, xTyPz → yPxPz, {xPzPy, xTyPz, zPxTy} → {yPxPz, xPzPy, zPyPx}, {yPzPx, xTyPz, zPxTy} → yPzPx, zPxTy → zPyPx, {xPyTz, xTyPz, xTyTz} → {yPxPz, xPyTz, yTzPx}, {zPxTy, xTzPy, xTyTz} → {zPyPx, yPxTz, xTzPy}, xTyTz → {yPxTz, yTzPx, xTyTz}.

Inspection of the above relationships shows that, if any individual voter elevates one alternative in his ordering, then the social ordering will either remain the same or elevate that alternative in the social ordering, and, therefore, the assertion is proven.

<u>Condition 3:</u> **The Independence of Irrelevant Alternatives**

Since by construction if xPy, any third stage solution reduces to xPy [i.e] If O(x,y) = xPy, then O(x,y) = O'(x,y) for all O and O'.

<u>Condition 4:</u> **Citizens' Sovereignty**

The social choice is imposed if there is some pair of alternatives x and y such that the social ordering will always be yPx even if, for every individual voter xP_iy. In the algorithm under consideration here, if the majority of voters prefers x to y, then xPy and vice versa by construction.

<u>Condition 5:</u> **The Condition of Nondictatorship**

There is no dictator by construction. If the majority prefers or is indifferent to x over y, then xQy and vice versa.

The proof considering both P and I is similar to the above.

Conclusions

Arrow allows for the existence of tied alternatives via his social choice function, C(S), which selects the most preferred alternative or set of alternatives from an ordering, but not for the existence of tied social orderings. However, as Arrow himself acknowledges, the Axiom of Completeness demands that "either xPy or yPx or both." Similarly, for three alternatives completeness would demand the consideration of orderings of the form xPyPz, yPzPx or both as well as combinations of all other possible social orderings. When tie social orderings are allowed as part of the range of a SWF, it can be shown that a rational SWF which is compliant with a strengthened version of Arrow's Axioms and Criteria is possible for the case of three alternatives.

These results can be extended to the general case of an arbitrary number of alternatives. We have demonstrated elsewhere an algorithm which provides solutions in the general case and shown that the solutions meet a strengthened version of Arrow's Axioms and Criteria. We have also proven that the general algorithm provides solutions for any number of alternatives and voter/consumers, and, therefore, that social choice is possible.

Appendix 1

Case	Binary Solutions	Ternary Solution
1	xPy, xPz, yPz	xPyPz
2	xPy, xPz, zPy	xPzPy
3	xPy, zPx, yPz	{xPyPz, yPzPx, zPxPy}
4	xPy, zPx, zPy	zPxPy
5	yPx, xPz, yPz	yPxPz
6	yPx, xPz, zPy	{yPxPz, xPzPy, zPyPx}
7	yPx, zPx, yPz	yPzPx
8	yPx, zPx, zPy	zPxPy
9	xPy, xPz, yIz	xPyIz
10	xPy, zPx, yIz	{zPxPy, xPyIz, yIzPx}
11	yPx, xPz, yIz	{yPxPz, xPyIz, yIzPx}
12	yPx, zPx, yIz	yIzPx
13	xPy, xIz, yPz	{xPyPz, yPxIz, xIzPy}
14	xPy, xIz, zPy	xIzPy
15	yPx, xIz, yPz	yPxIz
16	yPx, xIz, zPy	{zPyPx, yPxIz, xIzPy}
17	xIy, xPz, yPz	xIyPz
18	xIy, xPz, zPy	{xPzPy, xIyPz, zPxIy}
19	xIy, zPx, yPz	{yPzPx, xIyPz, zPxIy}
20	xIy, zPx, zPy	zPxIy
21	xPy, xIz, yIz	{xPyIz, xIzPy, xIyIz}
22	yPx, xIz, yIz	{yPxIz, yIzPx, xIyIz}
23	xIy, xPz, yIz	{xPyIz, xIyPz, xIyIz}
24	xIy, zPx, yIz	{zPxIy, yIzPx, xIyIz}
25	xIy, xIz, yPz	{yPxIz, xIyPz, xIyIz}
26	xIy, xIz, zPy	{zPxIy, xIzPy, xIyIz}

27	xIy, xIz, yIz	xIyIz
28	xPy, xPz, yTz	xPyTz
29	xPy, zPx, yTz	{zPxPy, xPyTz, yTzPx}
30	yPx, xPz, yTz	{yPxPz, xPyTz, yTzPx}
31	yPx, zPx, yTz	yTzPx
32	xPy, xTz, yPz	{xPyPz, yPxTz, xTzPy}
33	xPy, xTz, zPy	xTzPy
34	yPx, xTz, yPz	yPxTz
35	yPx, xTz, zPy	{zPyPx, yPxTz, xTzPy}
36	xTy, xPz, yPz	xTyPz
37	xTy, xPz, zPy	{xPzPy, xTyPz, zPxTy}
38	xTy, zPx, yPz	{yPzPx, xTyPz, zPxTy}
39	xTy, zPx, zPy	zPxTy
40	xPy, xTz, yTz	{xPyTz, xTzPy, xTyTz}
41	yPx, xTz, yTz	{yPxTz, yTzPx, xTyTz}
42	xTy, xPz, yTz	{xPyTz, xTyPz, xTyTz}
43	xTy, zPx, yTz	{zPxTy, yTzPx, xTyTz}
44	xTy, xTz, yPz	{yPxTz, xTyPz, xTyTz}
45	xTy, xTz, zPy	{zPxTy, xTzPy, xTyTz}
46	xTy, xTz, yTz	xTyTz
47	xPy, xIz, yTz	{xPyTz, yTzIx, zIxPy}
48	xPy, xTz, yIz	{xPyIz, yIzTx, zTxPy}
49	yPx, xIz, yTz	{yPxIz, xIzTy, zTyPx}
50	yPx, xTz, yIz	{yPxTz, xTzIy, zIyPx}
51	xIy, xPz, yTz	{xPzTy, zTyIx, yIxPz}
52	xIy, zPx, yTz	{zPxIy, xIyTz, yTzPx}
53	xTy, xPz, yIz	{xPzIy, zIyTx, yTxPz}
54	xTy, zPx, yIz	{zPxTy, xTyIz, yIzPx}
55	xIy, xTz, yPz	{yPzTx, zTxIy, xIyPz}

56	xIy, xTz, zPy	{zPyIx, yIxTz, xTzPy}
57	xTy, xIz, yPz	{yPzIx, zIxTy, xTyPz}
58	xTy, xIz, zPy	{zPyTx, yTxIz, xIzPy}
59	xIy, xIz, yTz	xIyTz
60	xIy, xTz, yIz	{xIyIz, yIzTx, zTxIy}
61	xTy, xIz, yIz	xTyIz
62	xIy, xTz, yTz	{xIyTz, yTzTx, zTxIy}
63	xTy, xIz, yTz	{xTyTz, yTzIx, zIxTy}
64	xTy, xTz, yIz	{xTyIz, yIzTx, zTxTy}

References

1. K. J. Arrow, “Social Choice and Individual Values,” John Wiley & Sons Inc., New York, 1951.
2. M. J. A. N. marquis de Condorcet, “Essai sur l'application de l'analyse á la probabilité des décisions rendues á la pluralité des voix”, Imprimerie Royale, Paris, 1785.
3. J. C. Lawrence, *The Possibility of Social Choice*, 1998 unpublished.
4. Y. Murakami, “Logic and Social Choice,” Routledge & Kegan Paul Ltd., London, 1968.
5. A. K. Sen, “Collective Choice and Social Welfare,” Holden-Day, San Francisco, 1970.